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Recent Developments in Integrable Systems and Related Topics of Mathematical Physics

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Preface

Our book is a selection of works presented at the Conference of Mathematical Physics “Kezenoi-Am 2016”. The Organising and Programme Committee of the conference tried to create a programme which embraces the variety of research directions inspired by the modern developments in Mathematical Physics and the theory of Integrable Systems. The authors of the included papers are well known mathematicians from several research groups in Europe and Russia. We hope that the book will attract the attention to these areas of research, and will be interesting both to experts and young researchers.

The Conference of Mathematical Physics “Kezenoi-Am 2016” is the first in the series of conferences which are being held partly in the city of Grozny, and partly at a beautiful mountain lake “Kezenoi-Am” of the Chechen Republic, Russia. These conferences are generously supported by the Chechen State University (CheSU), which is driven by the goal to support mathematical culture in Chechen republic. It is important to note that the Chairman of the Organizing Committee, Rector of the CheSU, Prof. Zaurbek Saidov, encourages the idea that the organization of international conferences with the participation of world recognized researches is the optimal way to motivate and attract students to research. During the participants' welcoming, he stressed out that he considers our conference to be a quite important step towards this direction. We are especially grateful to him for his support and help in the organization of this series of successful international conferences. We are also grateful to the Vice-rector of CheSU, Prof. Zaur Kindarov for his support and hospitality. Dr. Dmitry Grinev played an essential role in the organization of these conferences. We would like to thank him for his activity and patience throughout this procedure.

Finally, we acknowledge that this work was carried out within the framework of the State Programme of the Ministry of Education and Science of the Russian Federation, project 1.12873.2018/12.1.

Yaroslavl, Russia
June 2018

Victor M. Buchstaber
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Acronyms

BT	Bäcklund transformation
MI	Modulation instability
NLS	Nonlinear Schrödinger
NPE	Normal parabolic equation
PDE	Partial differential equation
PI	Painlevé I
PII	Painlevé II
QM	Quantum matrix
RE	Reflection equation
RW	Rogue wave
SSFM	Split-step Fourier method

Introduction

Victor M. Buchstaber[1], Sotiris Konstantinou-Rizos[2(✉)], and Alexander V. Mikhailov[3]

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Abstract. This chapter outlines the current trends in mathematical physics and the theory of integrable systems to which this book is devoted. The subject of this book is quite diverse and covers a wide range of problems. In fact, each chapter is devoted to a specific problem and contains links to articles and monographs in the literature where all the necessary definitions and constructions can be found. Here, we indicate the interrelationships between several research directions in this field of science that until recently were regarded as independent and distant from each other

Keywords: Integrable Systems · Mathematical Physics · Algebro-geometric methods

The modern development of Mathematical Physics has been greatly influenced by the theory of Integrable Systems. At the same time, the theory of Integrable Systems absorbed classical methods of Mathematical Physics based on the theory of differential equations, complex and functional analysis including the theory of distributions. Due to the theory of Integrable Systems, a variety of methods traditionally belonging to pure mathematics was able to solve concrete applied problems. In particular, algebro-geometric methods in periodical problems of soliton theory and the spectral theory of Schrödinger operators in periodic electromagnetic fields gave new life to algebraic geometry, the theory of Abelian functions, the theory of Riemann surfaces and the theory of functional equations. It is remarkable that methods of Integrable Systems were used in the solution of the long standing classical Riemann-Schottky problem in algebraic geometry. The theory of finite-dimensional Lie algebras plays an important role in classical mechanics. The theory of Integrable Systems stimulated further development of the theory of infinite-dimensional Lie algebras, and gave rise to a number of unsolved problems and fruitful concepts.

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It is well known that Mathematical Physics and the theory of Integrable Systems are closely related to classical Differential Geometry. Long standing problems, such as the embedding theorems and a description of Egorov's coordinates, were proved to be equivalent to key problems in the theory of Integrable Systems. The theory of Integrable Systems also creates new, useful and interesting research directions in the theory of infinite dimensional manifolds, which originates from problems in Hydrodynamics and the Optimal Control theory. The theory of Hopf algebras (a fundamental part of algebraic topology) creates the basis for the theory of quantum groups which, in turn, was motivated by the theory of Integrable Systems. Recently, the emerged and rapidly developing theory of Frobenius manifolds was proved to be connected with Hamiltonian systems of partial differential equations with small parameter. The development of the above-listed directions in classical and modern Mathematics, motivated by problems in Mathematical Physics and Theory of Integrable Systems, is far from completion.

As already mentioned in the preface, this book constitutes a selection of works presented at the Conference of Mathematical Physics "Kezenoi-Am 2016". In this selection, we aimed to reflect the variety of problems and approaches mentioned above. We have been pursuing the goal to foster the development of interfaces between traditionally disjoint areas of research.

Numerical Instability of the Akhmediev Breather and a Finite-Gap Model of It

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Abstract. The focusing Nonlinear Schrödinger (NLS) equation is the simplest universal model describing the modulation instability (MI) of quasi monochromatic waves in weakly nonlinear media, considered the main physical mechanism for the appearance of rogue (anomalous) waves (RWs) in Nature. In this paper we study the numerical instabilities of the Akhmediev breather, the simplest space periodic, one-mode perturbation of the unstable background, limiting our considerations to the simplest case of one unstable mode. In agreement with recent theoretical findings of the authors, in the situation in which the round-off errors are negligible with respect to the perturbations due to the discrete scheme used in the numerical experiments, the split-step Fourier method (SSFM), the numerical output is well-described by a suitable genus 2 finite-gap solution of NLS. This solution can be written in terms of different elementary functions in different time regions and, ultimately, it shows an exact recurrence of rogue waves described, at each appearance, by the Akhmediev breather. We discover a remarkable empirical formula connecting the recurrence time with the number of time steps used in the SSFM and, via our recent theoretical findings, we establish that the SSFM opens up a vertical unstable gap whose length can be computed with high accuracy, and is proportional to the inverse of the square of the number of time steps used in the SSFM. This neat picture essentially changes when the round-off error is sufficiently large. Indeed experiments in standard double precision show serious instabilities in both the periods and phases of the recurrence. In contrast with it, as predicted by the theory, replacing the exact Akhmediev Cauchy datum by its first harmonic approximation, we only slightly modify the numerical output.

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Let us also remark, that the first rogue wave appearance is completely stable in all experiments and is in perfect agreement with the Akhmediev formula and with the theoretical prediction in terms of the Cauchy data.

Keywords: Akhmediev breather · Rogue waves
Split-step Fourier method

1 Introduction

The self-focusing Nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \quad u = u(x,t) \in \mathbb{C} \tag{1}$$

is a universal model in the description of the propagation of a quasi monochromatic wave in a weakly nonlinear medium; in particular, it is relevant in deep water [59], in nonlinear optics [15,46,49], in Langmuir waves in a plasma [51], and in the theory of attracting Bose-Einstein condensates [14]. It is well-known that its elementary solution

$$u_0(x,t) = e^{2it}, \tag{2}$$

describing Stokes waves [50] in a water wave context, a state of constant light intensity in nonlinear optics, and a state of constant boson density in a Bose-Einstein condensate, is unstable under the perturbation of waves with sufficiently large wave length [11,47,52,53,59,64], and this modulation instability (MI) is considered as the main cause for the formation of rogue (anomalous, extreme, freak) waves (RWs) in Nature [21,26,31,32,43,44].

The integrable nature [60] of the NLS equation allows one to construct solutions corresponding to perturbations of the background by degenerating finite-gap solutions [10,29,35,36], when the spectral curve becomes rational, or, more directly, using classical Darboux [41], Dressing [61,62] techniques. Among these basic solutions, we mention the Peregrine soliton [45], rationally localized in x and t over the background (2), the so-called Kuznetsov [37] - Ma [40] soliton, exponentially localized in space over the background and periodic in time; the so-called Akhmediev breather [4–6], periodic in x and exponentially localized in time over the background (2). These solutions have also been generalized to the case of multi-soliton solutions, describing their nonlinear interaction, see, f.i., [20,27,29,30,63], and to the case of integrable multicomponent NLS equations [9,19].

The soliton solution over the background (2) playing a basic role in this paper is the Akhmediev breather

$$\begin{array}{l} A(x,t;\phi,X,T,\rho) \equiv e^{2it+i\rho}\frac{\cosh[\Sigma(\phi)(t-T)+2i\phi]+\sin(\phi)\cos[K(\phi)(x-X)]}{\cosh[\Sigma(\phi)(t-T)]-\sin(\phi)\cos[K(\phi)(x-X)]}, \\ K(\phi) = 2\cos\phi, \quad \Sigma(\phi) = 2\sin(2\phi), \end{array} \tag{3}$$

exact solution of (1) for all values of the 4 real parameters ϕ, X, T, ρ, changing the background by the multiplicative phase factor $e^{4i\phi}$:

$$A(x,t;\phi,X,T,\rho) \to e^{2it+i(\rho\pm 2\phi)}, \quad \text{as} \quad t \to \pm\infty,$$

and reaching the amplitude maximum at the point (X, T), with

$$|A(X, T; \phi, X, T, \rho)| = 1 + 2 \sin \phi.$$

Concerning the NLS Cauchy problems in which the initial condition consists of a perturbation of the exact background (2), if such a perturbation is localized, then slowly modulated periodic oscillations described by the elliptic solution of (1) play a relevant role in the longtime regime [12,13]. If the initial perturbation is x-periodic, numerical experiments and qualitative considerations prior to our recent works indicated that the solutions of (1) exhibit instead time recurrence [7,38,39,56–58], as well as numerically induced chaos [1–3], in which solutions of Akhmediev type seem to play a relevant role [16–18].

We have recently started a systematic study of the Cauchy problem on the segment $[0, L]$, with periodic boundary conditions, considering, as initial condition, a generic, smooth, periodic, zero average, small perturbation of (2)

$$\begin{aligned} &u(x, 0) = 1 + \varepsilon(x), \\ &\varepsilon(x + L) = \varepsilon(x), \quad ||\varepsilon(x)||_\infty = \varepsilon \ll 1, \quad \int_0^L \varepsilon(x) dx = 0. \end{aligned} \tag{4}$$

It is well-known that, if one perturbs the constant solution (2) by a small monochromatic perturbation $\delta u(x,t) = d_1(t)e^{ikx} + d_2(t)e^{-ikx}$, then in the leading order, the perturbation satisfies the linearized NLS equation

$$i\delta u_t + \delta u_{xx} + 4\delta u + 2e^{4it}\bar{\delta u} = 0, \tag{5}$$

and a direct calculation shows that its solution is unstable iff $|k| < 2$. This implies that, in our periodic Cauchy problem, if we expand the perturbation in Fourier series,

$$\varepsilon(x) = \sum_{j \geq 1} \left(c_j e^{ik_j x} + c_{-j} e^{-ik_j x} \right), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(\varepsilon), \tag{6}$$

the first M modes $\pm k_j$, $1 \leq j \leq M$, where $M \in \mathbb{N}^+$ is defined through the inequalities

$$\frac{L}{\pi} - 1 < M < \frac{L}{\pi}, \quad \pi < L, \tag{7}$$

are unstable, since they give rise to exponentially growing and decaying waves of amplitudes $O(\varepsilon e^{\pm \sigma_j t})$, where the growing factors σ_j are defined by

$$\sigma_j = k_j \sqrt{4 - k_j^2}, \quad 1 \leq j \leq M.$$

Therefore, the perturbation becomes $O(1)$ at times $T_j = O({\sigma_j}^{-1} |\log \ \varepsilon|)$, $1 \leq j \leq M$ (the first stage of MI), while the remaining modes give rise to oscillations of amplitude $O(\varepsilon e^{\pm i\omega_j t})$, where $\omega_j = k_j \sqrt{k_j^2 - 4}$, $\ j > M$, and therefore are stable.

Using the finite gap method [28,34,42], we have established that the leading part of the evolution of a generic periodic perturbation of the constant background is described by finite-gap solutions associated with hyperelliptic genus $2M$ Riemann surfaces, where M is the number of unstable modes. These solutions are well-approximated by an infinite time sequence of RWs, and the m-th RW of the sequence is described by the m-breather ($m \le M$) solutions of Akhmediev type, whose parameters vary at each appearance following a simple law in terms of the initial data [23–25]. In particular, in the simplest case of a single unstable mode $M = 1$, corresponding to the choice $\pi < L < 2\pi$, we have established that the initial condition

$$u(x,0) = 1 + c_1 \exp(ik_1 x) + c_{-1}\exp(-ik_1 x),\ \ |c_1|, |c_{-1}| = O(\varepsilon),\ \ k_1 = \frac{2\pi}{L} \quad (8)$$

evolves in the following way.

If $0 \le t \le O(1)$, we have the first linear stage of the MI, described by the linearized NLS around the background (2):

$$\begin{array}{l} u(x,t) = e^{2it}\Big\{1 + \frac{2}{\sigma_1}\Big[|\alpha_1|\cos\Big(k_1(x - X_1^+)\Big)e^{\sigma_1 t + i\phi_1} + \\ |\beta_1|\cos\Big(k_1(x - X_1^-)\Big)e^{-\sigma_1 t - i\phi_1}\Big]\Big\} + O(\varepsilon^2|\log\varepsilon|), \end{array} \quad (9)$$

where

$$\begin{array}{l} \alpha_1 = \overline{c_1} - e^{2i\phi_1} c_{-1}, \\ \beta_1 = \overline{c_{-1}} - e^{-2i\phi_1} c_1, \\ \phi_1 = \arccos\left(\frac{\pi}{L}\right) = \arccos\left(\frac{k_1}{2}\right), \end{array}$$

and $X_1^{\pm}$, defined as

$$X_1^+ = \frac{\arg(\alpha_1) - \phi_1 + \pi/2}{k_1}, \quad X_1^- = \frac{-\arg(\beta_1) - \phi_1 + \pi/2}{k_1},$$

are the positions of the maxima of the sinusoidal wave decomposition of the growing and decaying unstable modes. Therefore: *the initial datum splits into exponentially growing and decaying waves, respectively the* α*- and* β*-waves, each one carrying half of the information encoded in the initial datum.*

If $|t - T_1(|\alpha_1|)| \le O(1)$, where

$$T_1(\zeta) \equiv \frac{1}{\sigma_1}\log\left(\frac{(\sigma_1)^2}{2\zeta}\right), \quad \zeta > 0,$$

then

$$u(x,t) = A(x,t;\phi_1, X_1^+, T_1(|\alpha_1|), 2\phi_1) + O(\varepsilon), \quad (10)$$

where $A(x,t;\phi,X,T,\rho)$ is the Akhmediev breather (3). It follows that *the first RW appears in the time interval* $|t - T_1(|\alpha_1|)| \le O(1)$ *and is described by the the Akhmediev breather, whose parameters are expressed in terms of the initial data through elementary functions*. Such a RW, appearing about the logarithmically large time $T_1(|\alpha_1|) = O(\sigma_1^{-1}|\log\varepsilon|)$, is exponentially localized in an $O(1)$ time interval over the background u_0, changing it by the multiplicative phase factor $e^{4i\phi_1}$; in addition, the modulus of the first RW (10) has its maximum at $t = T_1(|\alpha_1|)$, at the point X_1^+, and the value of this maximum has the upper bound

$$|A(x,t;\phi_1,X_1^+,T_1(|\alpha_1|),2\phi_1)| = 1 + 2\sin\phi_1 < 1 + \sqrt{3} \sim 2.732,$$

2.732 times the background amplitude, consequence of the formula $\sin\phi_1 = \sqrt{1-(\pi/L)^2}$, $\pi < L < 2\pi$, and obtained when L is close to 2π. We notice that the position $x = X_1^+$ of the maximum of the RW coincides with the position of the maximum of the growing sinusoidal wave of the linearized theory; this is due to the absence of nonlinear interactions with other unstable modes. We finally remark that, *in the first appearance, the RW contains information, at the leading order, only on half of the initial wave, the half associated with the α-wave.*

It is easy to verify that the two representations (9) and (10) of the solution, valid respectively in the time intervals $0 \le t \le O(1)$ and $|t - T_1(|\alpha_1|)| \le O(1)$, have the same behavior

$$u(x,t) \sim e^{2it}\left(1 + \frac{|\alpha_1|}{\sin 2\phi_1} e^{\sigma_1 t + i\phi_1}\cos[k_1(x - X_1^+)]\right) \qquad (11)$$

in the intermediate region $O(1) \ll t \ll T_1(|\alpha_1|)$; therefore they match successfully.

The periodicity properties of the θ-function representation of the solution [28] imply that, *whithin $O(\varepsilon^2|\log\varepsilon|)$ corrections, the solution of this Cauchy problem is also periodic in t, with period T_p, up to the multiplicative phase factor exp $(2iT_p + 4i\phi_1)$ and up to the global x-translation of the quantity Δ_x:*

$$u(x,t+T_p) = e^{2iT_p+4i\phi_1}u(x-\Delta_x,t) + O(\varepsilon^2|\log\varepsilon|),$$

where

$$\begin{aligned} T_p &= T_1(|\alpha_1|) + T_1(|\beta_1|) = \tfrac{2}{\sigma_1}\log\left(\tfrac{\sigma_1^2}{2\sqrt{|\alpha_1\beta_1|}}\right) = O(2\sigma_1^{-1}|\log\varepsilon|), \\ \Delta_x &= X_1^+ - X_1^- = \tfrac{\arg(\alpha_1\beta_1)}{k_1}. \end{aligned} \qquad (12)$$

The time periodicity allows one to infer that *the above Cauchy problem leads to an exact recurrence of RWs (of the nonlinear stages of MI), alternating with an exact recurrence of linear stages of MI* [23]. We have, in particular, the following RW sequence.

This Cauchy problem gives rise to an infinite sequence of RWs, and the m^{th} RW of the sequence ($m \ge 1$) is described, in the time interval $|t - T_1(|\alpha_1|) - (m-1)T_p| \le O(1)$, by the analytic deterministic formula:

$$u(x,t) = A\left(x,t;\phi_1,x_1^{(m)},t_1^{(m)},\rho^{(m)}\right) + O(\varepsilon), \quad m \ge 1, \qquad (13)$$

where

$$\begin{aligned} x_1^{(m)} &= X_1^+ + (m-1)\Delta^{(x)}, \quad t_1^{(m)} = T_1(|\alpha_1|) + (m-1)T_p, \\ \rho^{(m)} &= 2\phi_1 + (m-1)4\phi_1, \end{aligned} \qquad (14)$$

in terms of the initial data [23] *(see Figs.* 1 *and* 2*). Apart from the first RW appearance, in which the RW contains information only on half of the initial data (the one encoded in the parameter* α_1*), in all the subsequent appearances the RW contains, at the leading order, informations on the full unstable part of the initial datum, encoded in both parameters* α_1 *and* β_1*.*

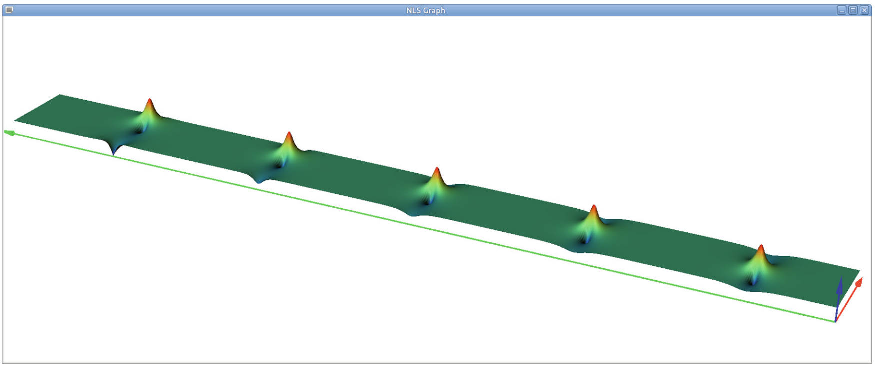

Fig. 1. The 3D plotting of $|u(x,t)|$ describing the RW sequence, obtained through the numerical integration of NLS via the Split Step Fourier Method (SSFM) [8,54,55]. Here $L = 6$ $(M = 1)$, with $c_1 = \varepsilon/2,\ c_{-1} = \varepsilon(0.3 - 0.4i)/2,\ \varepsilon = 10^{-4}$, and the short axis is the x-axis, with $x \in [-L/2, L/2]$. The numerical output is in perfect agreement with the theoretical predictions.

Four remarks are important at this point, in addition to the considerations on the instabilities we made at the beginning of this section.

(a) If the initial condition (8) is replaced by a more general initial condition (4), (6) in which we excite also all the stable modes, then (9) is replaced by a formula containing also the infinitely many $O(\varepsilon)$ oscillations corresponding to the stable modes. But the behavior of the solution in the overlapping region $1 \ll |t| \ll O(\sigma_1^{-1}|\log\varepsilon|)$ is still given by (11) and matching at $O(1)$ is not affected. Therefore the sequence of RWs is still described by Eqs. (13), (14), and the differences between the two Cauchy problems are hidden in the $O(\varepsilon)$ corrections. *As far as the* $O(1)$ *RW recurrence is concerned, only the part of the initial perturbation* $\varepsilon(x)$ *exciting the unstable mode is relevant.*

(b) The above results are valid up to $O(\varepsilon^2|\log\varepsilon|)$. It means that, in principle, the above RW recurrence formulae may not give a correct description for large times of $O((\varepsilon|\log\varepsilon|)^{-1})$; but since $O((\varepsilon|\log\varepsilon|)^{-1})$ is much larger than the recurrence time $O(|\log\varepsilon|)$, it follows that the above formulae should give an accurate description of the RW recurrence for many consecutive appearances of the RWs.

(c) Since the solution of the Cauchy problem is ultimately described by different elementary functions in different time intervals of the positive time axis, and since these different representations obviously match in their overlapping time

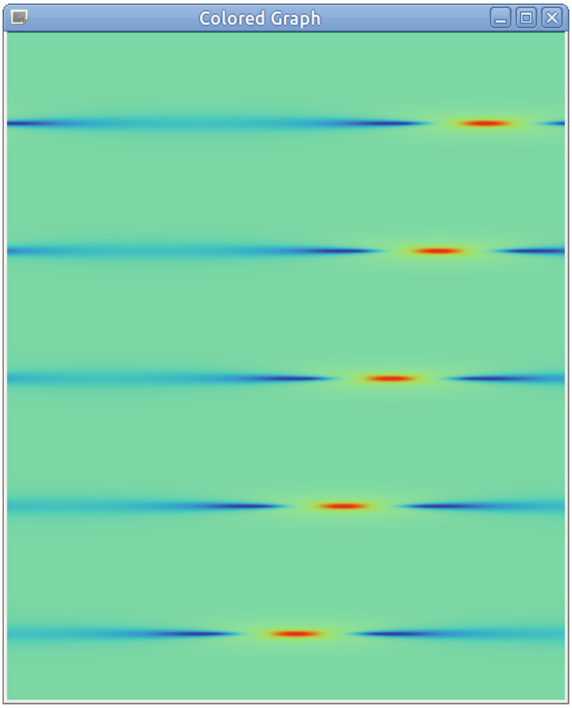

Fig. 2. The color level plotting for the numerical experiment of Fig. 1, in which the periodicity properties of the dynamics are evident.

regions, these finite gap results naturally motivate the introduction of a matched asymptotic expansions (MAE) approach, presented in the paper [24]. MAEs works well if $M = 1$, but can deal with a finite number $M > 1$ of unstable modes only under very special initial data.

(d) We remark that recurrence phenomena were recently observed experimentally in a super water wave tank. The particular semi-period phase shift was interpreted as the effect of small dissipation, as confirmed by numerical experiments (see [33]).

Since the above considerations establish the theoretical relevance of the Akhmediev breather in the description of each RW appearance in the time sequence, a natural and interesting open problem is the study of the numerical and physical instabilities of the Akhmediev breather.

In this paper we investigate experimentally the numerical instabilities, limiting our considerations to the simplest case of one unstable mode $M = 1$. In agreement with our recent theoretical findings, in the situation in which the round-off errors are negligible with respect to the perturbations due to the discrete scheme used in the numerical experiments: the Split-Step Fourier Method (SSFM) [8,54,55], the numerical output is well-described by a suitable genus 2 finite-gap solution of NLS. This solution can be written in terms of different elementary functions in different time regions and, ultimately, it shows an exact recurrence of rogue waves described by the Akhmediev breather whose parameters, different at each appearance, are given in terms of the initial data via elementary functions. We discover a remarkable empirical formula connecting the recurrence time with the number of time steps used in the SSFM and, via our recent theoretical findings in [23], we establish that the SSFM opens up a vertical unstable gap whose length can be computed with high accuracy, and is proportional to the inverse of the square of the number of time steps used in the SSFM. This neat picture essentially changes when the round-off error is sufficiently large. Indeed experiments in standard double precision show serious instabilities in both the periods and phases of the recurrence. In contrast with it, as predicted by the theory, replacing the exact Akhmediev Cauchy datum by its first harmonic approximation, we only slightly modify the numerical output. Let us also remark that the first rogue wave appearance is completely stable in all experiments, in perfect agreement with the Akhmediev formula and with the theoretical predictions [23] in terms of the Cauchy data.

2 A Short Summary of Finite Gap Results

Here we summarize the classical and recent results on the NLS finite gap theory used in this paper. The interested reader can find additional details in [23].

The zero-curvature representation of the NLS equation (1) is given by the following pair of linear problems [60]:

$$\mathbf{\Psi}_x(\lambda,x,t) = U(\lambda,x,t)\mathbf{\Psi}(\lambda,x,t), \tag{15}$$

$$\mathbf{\Psi}_t(\lambda,x,t) = V(\lambda,x,t)\mathbf{\Psi}(\lambda,x,t), \tag{16}$$

$$U = \begin{bmatrix} -i\lambda & iu(x,t) \\ i\overline{u(x,t)} & i\lambda \end{bmatrix},$$

$$V(\lambda,x,t) = \begin{bmatrix} -2i\lambda^2 + iu(x,t)\overline{u(x,t)} & 2i\lambda u(x,t) - u_x(x,t) \\ 2i\lambda\overline{u(x,t)} + \overline{u_x(x,t)} & 2i\lambda^2 - iu(x,t)\overline{u(x,t)} \end{bmatrix},$$

where

$$\mathbf{\Psi}(\lambda,x,t) = \begin{bmatrix} \mathit{\Psi}_1(\lambda,x,t) \\ \mathit{\Psi}_2(\lambda,x,t) \end{bmatrix}.$$

In the x-periodic problem with period L, we have the **main spectrum** and the **auxiliary spectrum**. If $\mathit{\Psi}(\lambda,x,t)$ is a fundamental matrix solution of (15), (16) such that $\mathit{\Psi}(\lambda,0,0)$ is the identity, then the monodromy matrix $\hat{T}(\lambda)$ is defined by: $\hat{T}(\lambda) = \mathit{\Psi}(\lambda,L,0)$. The eigenvalues and eigenvectors of $\hat{T}(\lambda)$ are defined on a two-sheeted covering of the λ-plane. This Riemann surface $\mathit{\Gamma}$ is called the **spectral curve** and does not depend on time. The eigenvectors of $\hat{T}(\lambda)$ are the Bloch eigenfunctions

$$\begin{gathered} \mathbf{\Psi}_x(\gamma,x,t) = U(\lambda(\gamma),x,t)\mathbf{\Psi}(\gamma,x,t), \\ \mathbf{\Psi}(\gamma,x+L,t) = e^{iLp(\gamma)}\mathbf{\Psi}(\gamma,x,t), \gamma \in \mathit{\Gamma}, \end{gathered}$$

$\lambda(\gamma)$ denote the projection of the point γ to the λ-plane.

The spectrum is exactly the projection of the set $\{\gamma \in \Gamma, \operatorname{Im} p(\gamma) = 0\}$ to the λ-plane. The end points of the spectrum are the branch points and the double points (obtained merging pairs of branch points) of Γ, at which $e^{iLp(\gamma)} = \pm 1$:

$$\mathbf{\Psi}(\gamma, x + L, t) = \pm \mathbf{\Psi}(\gamma, x, t), \quad \gamma \in \Gamma.$$

A potential $u(x,t)$ is called **finite-gap** if the spectral curve Γ is algebraic; i.e., if it can be written in the form

$$\nu^2 = \prod_{j=1}^{2g+2} (\lambda - E_j).$$

It means that Γ has only a finite number of branch points and non-removable double points. These potentials can be written in terms of Riemann theta-functions [28]. Any smooth, periodic in x solution admits an arbitrarily good finite gap approximation, for any fixed time interval.

The auxiliary spectrum can be defined as the set of zeroes of the first component of the Bloch eigenfunction: $\Psi_1(\gamma, x, t) = 0$, therefore it is called **divisor of zeroes.** The zeroes of $\Psi_1(\gamma, x, t)$ depend on x and t, and the x and t dynamics become linear after the Abel transform.

The spectral curve Γ_0 corresponding to the background (2) is rational, and a point $\gamma \in \Gamma_0$ is a pair of complex numbers $\gamma = (\lambda, \mu)$ satisfying the quadratic equation $\mu^2 = \lambda^2 + 1$.

The Bloch eigenfunctions can be easily calculated explicitly:

$$\psi^{\pm}(\gamma, x) = \begin{bmatrix} 1 \\ \lambda(\gamma) \pm \mu(\gamma) \end{bmatrix} e^{\pm i\mu(\gamma)x},$$

and are periodic (antiperiodic) iff $\frac{L}{2\pi}\mu$ is an even (an odd) integer. Let us introduce the following enumeration of the periodic and antiperiodic spectral points:

$$\mu_n = \tfrac{\pi n}{L} = \tfrac{k_n}{2}, \quad \lambda_n^{\pm} = \pm\sqrt{\mu_n^2 - 1}, \quad n \in \mathbb{N}.$$

The curve Γ_0 has two branch points $E_0 = i$, $\overline{E_0} = -i$ corresponding to $n = 0$. If $n > 0$, we have only double points. The first M, such that $|\mu_n| < 1, \ 1 \le n \le M$, are unstable, where M is defined by (7); correspondingly $\lambda_n^{\pm}$ are pure imaginary and one can introduce the following convenient parametrization

$$\mu_n = \cos\phi_n, \quad \lambda_n^{\pm} = \pm i\sqrt{1 - \mu_n^2} = i \sin\phi_n, \quad 1 \le n \le M,$$

implying that

$$k_n = 2\cos\phi_n, \quad \sigma_n = 2\sin 2\phi_n, \quad 1 \le n \le M.$$

The remaining modes are such that $|\mu_n| > 1, \ n > M$ and are stable, and the corresponding $\lambda_n' s$ are real. In addition, the divisor of the unperturbed problem is located at the double points.

An initial $O(\varepsilon)$ perturbation of the type (4) perturbs this picture. The branch points $\lambda_0^{\pm}$ become $E_0 = i + O(\varepsilon^2)$ and $\bar{E}_0 = -i + O(\varepsilon^2)$, and all double points generically split into a pair of square root branch points, generating infinitely many gaps. If $1 \leq n \leq M$, λ_n^+ splits into the pair of branch points (E_{2n-1}, E_{2n}), and λ_n^- into the pair of branch points $(\bar{E}_{2n-1}, \bar{E}_{2n})$; if $n > M$, each λ_n splits into a pair of complex conjugate eigenvalues. In the simplest case in which $M = 1$ and one excites only the unstable mode as in (8), E_1, E_2 are the branch points obtained through the splitting of the excited mode $\lambda_1 = i \sin\phi_1$, and [23]

$$E_1 - E_2 = \frac{\sqrt{\alpha_1 \beta_1}}{\lambda_1^+} + O(\varepsilon^2), \tag{17}$$

while the infinitely many gaps associated with the stable modes are $O(\varepsilon^n)$, $n > 1$; therefore they give a negligible contribution to the solution of the Cauchy problem and can be erased from the picture.

The initial position of the divisor points $\gamma_{\pm 1}$, associated with the unstable modes $\lambda_1^{\pm}$, are [23]

$$\begin{aligned}
\lambda(\gamma_1) &= \lambda_1 + \frac{1}{4\lambda_1}\left[\alpha_1 + \beta_1\right] + O(\epsilon^2), \\
\lambda(\gamma_{-1}) &= -\lambda_1 + \frac{1}{4\lambda_1}\left[e^{2i\phi_1}\bar{\alpha}_1 + e^{-2i\phi_1}\bar{\beta}_1\right] + O(\epsilon^2).
\end{aligned}$$

Switching on time, the divisor points $\lambda(\gamma_{\pm 1})$ start moving in time and, after the period T_p defined in (12), they replace each other.

We end this section remarking that, from formulas (12) and (17) discovered in [23], one can write the following relations, at leading order, between the unstable gap details and the RW recurrence period T_p and x-shift Δ_x discussed in the previous section:

$$\begin{aligned}
T_p &= \tfrac{2}{\sigma_1} \log\left(\tfrac{\sigma_1^2}{2(\operatorname{Im}\lambda_1)|E_1 - E_2|}\right), \\
\Delta_x &= \tfrac{\arg(-(E_1 - E_2)^2)}{k_1}.
\end{aligned} \tag{18}$$

3 The Numerical Instability of the Akhmediev Breather

Due to the above instabilities, every time we have theoretical formulas describing time evolutions over the unstable background, it is important to test their stability under perturbations. Indeed: (a) in any numerical experiment one uses numerical schemes approximating NLS; in addition, round off errors are not avoidable. All these facts are expected to cause the opening of basically all gaps and, due to the instability, no matter how small are the gaps associated with the unstable modes, they will cause $O(1)$ effects during the evolution (see also [1,2,16]). (b) In physical phenomena involving weakly nonlinear quasi monochromatic waves, NLS is a first approximation of the reality, and higher order corrections have the effect of opening again all gaps, with $O(1)$ effects during the evolution caused by the unstable ones. At last, in a real experiment, a monochromatic initial perturbation like (8) should be replaced by a quasi-monochromatic approximation of it, often with random coefficients, opening again all the gaps associated with the unstable modes, with $O(1)$ effects during the evolution. The genericity of the Cauchy problems investigated in [23,24] and the associated numerical experiments seem to imply that the RW recurrences, analytically described in the previous section by a sequence of Akhmediev breathers, are expected to be stable under the above perturbations.

An interesting problem is to understand if, choosing instead at $t = 0$ the highly non generic Akhmediev breather (3) (we assume that $T > 0$ in (3)) as initial datum, numerical or physical perturbations yield $O(1)$ changes in the evolution with respect to the theoretical expectation (3). In this paper we concentrate on the instabilities generated by numerics, in the simplest case of a single unstable mode, postponing to a subsequent paper the study of instabilities due to the physical perturbations of the NLS equation.

The Akhmediev initial condition corresponds to a very special perturbation generating zero splitting for all the unstable and stable resonant points, and, in agreement with (18), it corresponds to $T_p = \infty$. Indeed, the Akhmediev breather (3) describes the appearance of a RW only in the time interval $|t - T| \le O(1)$. In addition, for the Akhmediev solution, $\alpha_1 = O(\varepsilon)$, and $|\beta_1| \ll \varepsilon$ (but $\beta_1 \neq 0$). In this non generic case, formula (17) is not valid, since we have an exact compensation between the leading term and the correction.

According to the above considerations, numerics introduces small perturbations to this non generic picture, opening small gaps for all stable and unstable modes. Again we erase from the picture the infinitely many stable gaps, and we are left with the gaps associated with the unstable mode λ_1^+ and with its complex conjugate. Therefore the numerical perturbation of the Akhmediev breather generates the finite gap configuration described in [23] and summarized in the previous section, and one expects that the genus 2 solution constructed there and describing the exact recurrence (13), (14) of RWs be the analytical model for the numerical instability of the Akhmediev breather. In some symmetrical cases these genus 2 solutions can be written in terms of elliptic functions [48].

Let us point out that, if we talk about numerical perturbations, we should distinguish two different sources: the difference between the continuous NLS model and the discretization used in the numerical scheme, and the round-off errors due to the finite number of digits used in the computations.

To study numerical instabilities, we used as numerical integrator the Split-Step Fourier Method (SSFM), also known as the split-step spectral method [8,54,55] [1]. It uses the following algorithm. First of all, the x-boundary conditions are chosen to be periodic with period L. Then one introduces a regular Cartesian lattice in the x,t-plane with the steps $\delta x = L/N_x$ and δt. Then, at each basic time step δt, the NLS evolution is split into linear and non-linear parts, which are executed subsequently (each basic step is split into two steps in the asymmetric version, or three steps in the symmetric version).

[1] The authors are very grateful to M. Sommacal for introducing us to this method and providing us with his personalized MatLab code.

Let us describe the basic step of the asymmetric version of the algorithm. Assume that the function $u(x_m,t_n)$, $x_m=-\frac{L}{2}+\frac{L_m}{N_x}$, $m=0,\dots,N_x-1$ is known. Then, we calculate $u(x_m,t_{n+1})$, $t_{n+1}=t_n+\delta t$ using the following sequence of operations:

1. We calculate an auxiliary function $u_1(x_m,t_n)$ by applying the non-linear step (here the dispersive term is omitted):

$$u_1(x_m,t_n)=u(x_m,t_n)\exp(2i|u(x_m,t_n)|^2\delta t). \tag{19}$$

2. We make the linear step in the momenta space (here the non-linear term is omitted).
   a. We apply the discrete Fourier transform to the function $u_1(x_m,t_n)$

$$\hat{u}_1(\hat{m},t_n)=\frac{1}{\sqrt{N_x}}\sum_{m=0}^{N_x-1}u(x_m,t_n)e^{\frac{-2\pi i}{N_x}\hat{m}(m-\frac{N_x}{2})},\quad \hat{m}=0,\dots,N_x-1. \tag{20}$$

   b. We define the discrete momentum $p_{\hat{m}}$ by

$$p_{\hat{m}}=\begin{cases}\frac{2\pi}{L}\hat{m} & \text{if} \quad \hat{m}\le\frac{N_x}{2}\\ \frac{2\pi}{L}(\hat{m}-N_x) & \text{if} \quad \hat{m}>\frac{N_x}{2}\end{cases} \tag{21}$$

   c. We make the linear step and define a second auxiliary function $\hat{u}_2(\hat{m},t_n)$

$$\hat{u}_2(\hat{m},t_n)=\hat{u}_1(\hat{m},t_n)\exp(-ip_{\hat{m}}^2\delta t). \tag{22}$$

3. We define the solution $u(x_m,t_n+1)$ at the next time step as the inverse Fourier transform of the function $\hat{u}_2(\hat{m},t_n)$:

$$u(x_m,t_{n+1})=\frac{1}{\sqrt{N_x}}\sum_{m=0}^{N_x-1}\hat{u}_2(\hat{m},t_n)e^{\frac{2\pi i}{N_x}\hat{m}(m-\frac{N_x}{2})},\quad m=0,\dots,N_x-1. \tag{23}$$

To perform the direct and the inverse Fourier transforms we use the Fast Fourier Transform algorithm implementation from the “Fastest Fourier Transform in the West” FFTW library [22].

Our goal was to see how well the Akhmediev solution can be reproduced in numerical experiments. We used the following settings:

1. We chose the simplest possible settings of the problem, with exactly one unstable mode ($\pi<L<2\pi$). We fixed $L=6$.
2. We used $N_x=512$. We made some attempts to modify N_x, and it appears that this change does not essentially affect the picture.
3. We chose the initial perturbation of order 10^{-4}. The first appearance time T_1 in our experiments was $\sim$4–6.
4. We chose the global time interval $T_{max}=60$, which is one order of magnitude greater then the first appearance time.

5. We used as time step in the integration procedure $\delta t = \frac{1}{10N}$, where N was varied from 50 to 15810.

To compare the effect of the round-off error with the effect of the numerical scheme discretization, we repeated some experiments twice, using C++ double and C++ quadruple precision. The typical round-off is 10^{-17} for double precision and 10^{-34} for quadruple precision; therefore, in the last case, it can be neglected.

Since the theory predicts that, to the leading order, the time evolution is determined by the excited unstable mode, i.e. by the first harmonics of the perturbation, we verified this prediction numerically.

4 Numerical Experiments

4.1 Effect of the Time Step

In the first series of numerical experiments, we studied how the quality of the approximation of the Akhmediev solution depends on δt. In all these experiments $L = 6$, $T_{max} = 60$, $N_x = 512$, $\delta t = 1/(10N)$. The Cauchy datum is the Akhmediev soliton (3) at $t = 0$, with $T = 5.8738$. All experiments were proceeded with quadruple precision.

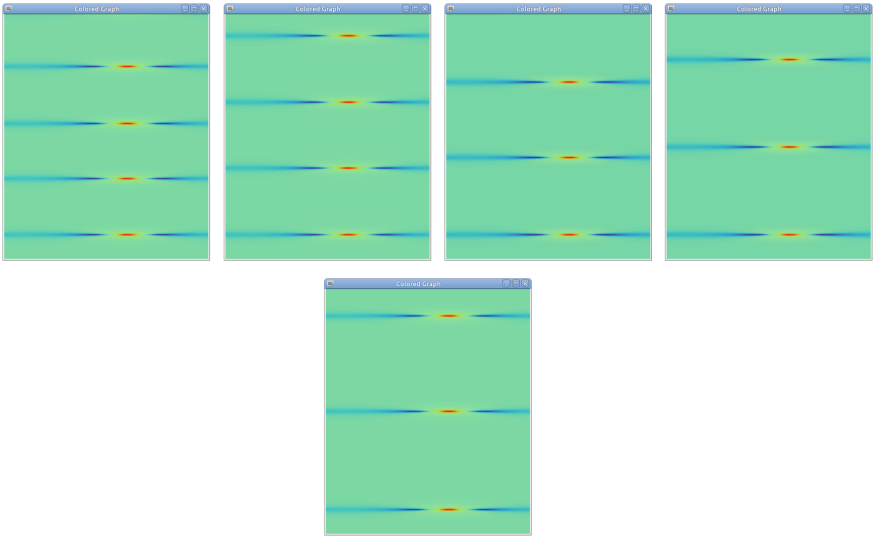

Fig. 3. Five level plots corresponding to $N = 158, 500.1581, 5000, 15810$ respectively.

In all these experiments, the numerical output shows the recurrence phenomena predicted by the finite-gap formulas, and not described by the exact Akhmediev solution. The position of the maxima at all appearances remains constant up to the grid step. According to the second formula in (18) it means that the perturbation due to the SSFM discretization opens a vertical gap. The size of the gap can be estimated using the first formula in (18):

$$|E_1 - E_2| = \frac{\sigma_1^2}{2\,\mathrm{Im}\,\lambda_1} e^{-\frac{\sigma_1 T_p}{2}}. \tag{24}$$

In the numerical experiments we assume $T_p = T_2 - T_1$, where T_1 and T_2 are respectively the times of the first and second appearances. In the experiments the second and the third recurrence times do not sensibly change. The results of the experiments are presented in the following table:

Experimental data				Finite-gap interpretation	
N	$T_2 - T_1$	$\exp\left(-\frac{\sigma_1(T_2-T_1)}{2}\right)$	$N^2\exp\left(-\frac{\sigma_1(T_2-T_1)}{2}\right)$	$\lvert E_1 - E_2\rvert$	$N^2\lvert E_1 - E_2\rvert$
158	13.7	$4.9 \cdot 10^{-6}$	0.123	$9.19 \cdot 10^{-6}$	0.229
500	16.3	$4.8 \cdot 10^{-7}$	0.120	$9.03 \cdot 10^{-7}$	0.226
1581	18.9	$4.8 \cdot 10^{-8}$	0.119	$8.88 \cdot 10^{-8}$	0.222
5000	21.4	$5.1 \cdot 10^{-9}$	0.128	$9.54 \cdot 10^{-9}$	0.239
15810	24.0	$5.0 \cdot 10^{-10}$	0.126	$9.38 \cdot 10^{-10}$	0.235

We see that the combination $N^2 \exp\left(-\frac{\sigma_1(T_2-T_1)}{2}\right)$ remains approximately constant, implying the following empirical law relating the recurrence time with the time step in the SSFM:

$$T_2 - T_1 \sim \frac{2}{\sigma_1} \log\left(\frac{N^2}{0.125}\right) \tag{25}$$

It is an open interesting problem in numerical analysis to explain this observation analytically.

We see that the recurrence in numerical solutions is well described by the finite-gap solutions. Thanks to Formula (24), one can estimate the size of the gap opened by the numerical perturbation, and the fact that such size is proportional to $1/N^2$ (see Table):

$$|E_1 - E_2| \sim \frac{0.125\ \sigma_1^2}{2\,\mathrm{Im}\,\lambda_1} \frac{1}{N^2}. \tag{26}$$

We conclude that the numerical output corresponding to the Akhmediev initial conditions is well described by the analytic formulas (13), (14), where

$$T_1(|\alpha_1|) = T_1, \quad \Delta_x = 0, \quad T_p = T_2 - T_1 \text{ given by } (25).$$

For completeness, we also provide two 3D plots of the numerical solutions corresponding to the extreme values of N: $N = 158$ and $N = 15810$.

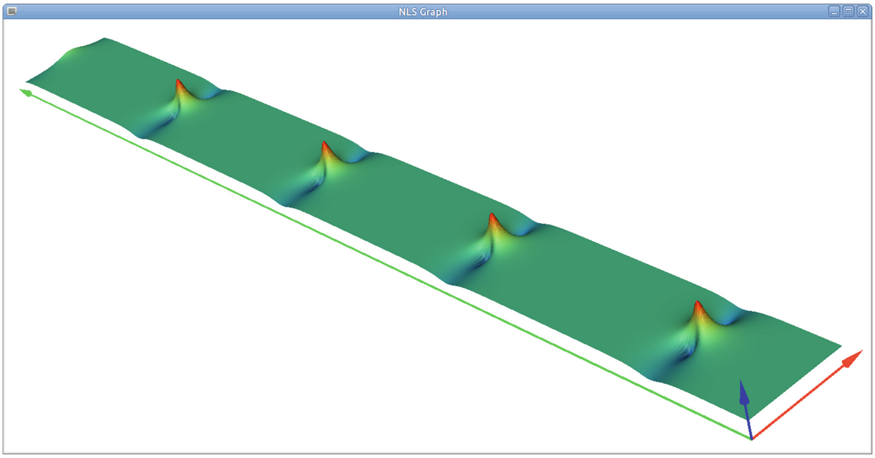

Fig. 4. $N = 158$. First appearance time = 5.8734, maximum of $|u|$ at the first appearance = 2.7039, position of the maximum = 0.527, recurrence times between consecutive appearances: 13.73, 13.49, 13.97 respectively

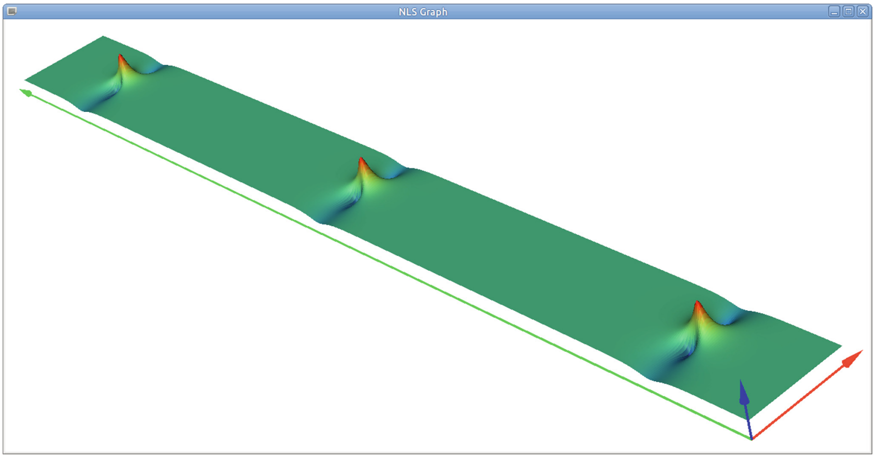

Fig. 5. $N = 15810$. First appearance time = 5.8738, maximum of $|u|$ at the first appearance = 2.7039, position of the maximaum = 0.527, recurrence times between consecutive appearances: 24.0, 23.4 respectively

4.2 Effect of Round-Off Errors

To study the effect of the round-off errors, we repeated the above experiments with double precision.

We see that, for $N = 500$, the round-off errors do not change dramatically the recurrence times, but the spatial positions of the maxima have a notable change after few recurrences. In the right pictures we see that, for $N = 5000$,

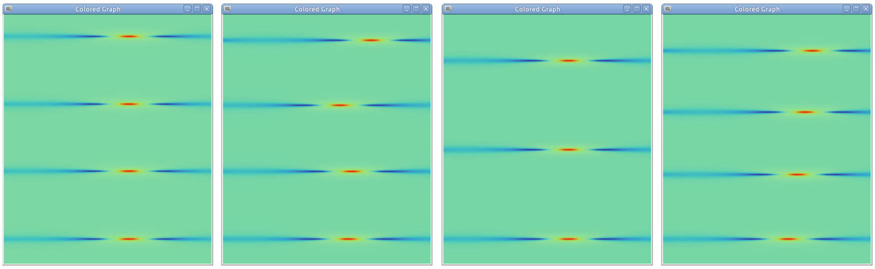

Fig. 6. The left pair of pictures shows the $N = 500$ calculations (left for quadruple precision and right for double precision). The right pair of pictures shows the $N = 5000$ calculations (left for quadruple precision and right for double precision).

the round-off error dramatically changes the recurrence time. We conclude that, in double precision experiments with sufficiently large N, the numerical perturbation due to round-off becomes more relevant than the numerical perturbation due to numerical scheme.

To illustrate it we also provide three numerical experiments made with double precision and time steps of the same order of magnitude.

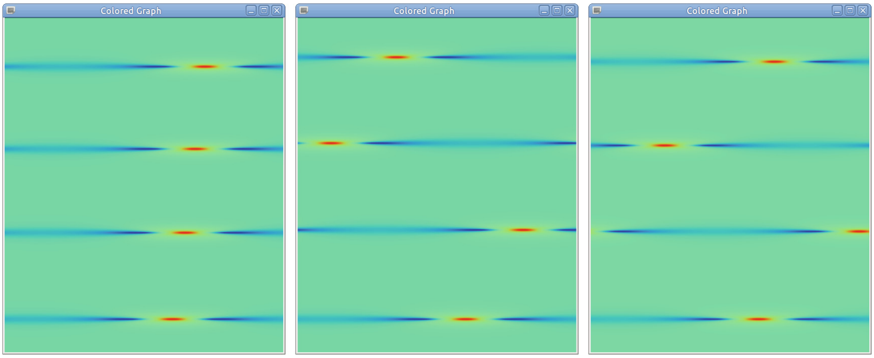

Fig. 7. Level plots for $N = 5000$, $N = 7500$ and $N = 10000$ respectively.

We see that the finite-gap approximation is still relevant, but the lengths and the orientation of the gap become more or less random. In fact, increasing the number of steps, we increase the influence of the round-off error.

4.3 Effect of the Cauchy Data

As it was shown in [23] theoretically, small stable harmonics of the Cauchy data do not seriously affect the leading order approximation. In this Section we verify that this situation is also relevant in numerical experiments. To do it, we check how stable is the numerical output when one replaces the exact Akhmediev initial condition by its unstable part, containing only the zero and the first harmonics:

$$u(x,0) = 1 + c_1 e^{ik_1 x} + c_{-1} e^{-ik_1 x},$$

where the coefficients c_1, c_{-1} are the first harmonics Fourier coefficients of the Akhmediev initial data. In our experiments

$$c_1 = 0.22341792182984515786378155403997297 \cdot 10^{-4} + 0.11311151504280589935931368075404486 \cdot 10^{-4} i,$$

$$c_{-1} = -1.7601617675954215179181727840739775 23 \cdot 10^{-14} + 0.25041921377970522403602105355224597 65 \cdot 10^{-4} i;$$

(we slightly corrected them to have $\beta_1 = 0$ up to the round-off error).

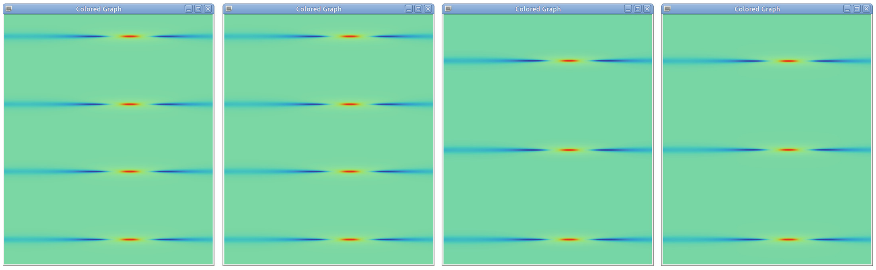

Fig. 8. The left pair of pictures shows the $N = 500$ experiments (the left one for the Akhmediev Cauchy datum and right one for the Cauchy datum containing only the first harmonics of the Akhmediev solution). The right pair of pictures shows the $N = 5000$ experiments (the left one for the Akhmediev Cauchy datum and the right one for the Cauchy datum containing only the first harmonics of the Akhmediev solution). Both numerical experiments were made with quadruple precision.

We see that, also for high-precision calculations, the difference between these two numerical outputs is very small (much less relevant than the double precision round-off error in the previous Section).

4.4 Effect of the x-Grid Size

To estimate the effect of the spatial discretization size on the numerical simulations, we made the following experiment: we took $N = 5000$ and chose quadruple precision (from the previous experiments, it corresponds to a very high accuracy), and we repeated the above experiments with N_x reduced 4 times: $N_x = 128$. At the level of the 3D output as well as at the level of the recurrence times and phase shift, the effect was negligible. We do not show the corresponding pictures because the difference is not visible. Therefore we conclude that the size of the space discretization does not play an important role in this study. The theoretical explanation in clearly due to the fact that the Akhmediev solution is very smooth for all t; therefore the higher harmonics are extremely small.

5 Conclusions

In this paper we have studied the numerical instabilities of the Akhmediev exact solution of the self-focusing Nonlinear Schrödinger equation, describing the simplest one-mode, x-periodic perturbation of the unstable constant background solution, limiting our considerations to the simplest case of one unstable mode. In agreement with the theoretical predictions associated with the theory developed in [23], in the situation in which the round-off errors are negligible with respect to the perturbations due to the discrete numerical scheme, the numerical output shows that the Akhmediev breather is unstable, and that this instability is well-described by genus 2 finite-gap solutions. These solutions are well-approximated by different elementary functions in different time regions, describing a time sequence of Akhmediev one-breathers.

We discover the remarkable formulas (25), (26) connecting the recurrence time and the gap opening to the number of time steps of the SSFM used as the numerical scheme. In particular, the length of the two gaps opened by the SSFM is proportional to the inverse of the square of the number of time steps. Since the RW sequence generated by the numerical scheme has no phase shifts, it follows that that these gaps are open vertically $\mathrm{Re}\, E_1 = \mathrm{Re}\, E_2 = 0$.

This clean picture essentially changes when the round-off error is sufficiently large. Indeed, the standard double precision experiments show serious instabilities in both periods and phases of the recurrence. In particular, increasing the number of time steps, we increase the instability. In contrast with it, replacing the exact Akhmediev Cauchy datum by the first harmonic approximation, we only slightly modify the numerical output, as predicted by the theory.

Let us also remark that the first appearance time, the position of the maximum and the value of it are completely stable in all experiments, and in perfect agreement with the Akhmediev formula, as well as with the theoretical predictions coming from [23] in terms of the Cauchy data.

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Movable Poles of Painlevé I Transcendents and Singularities of Monodromy Data Manifolds

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Abstract. We consider a classification of solutions to the first Painlevé equation with respect to distribution of their poles at infinity. A connection is found between singularities of two-dimensional monodromy data manifold and analytic properties of solutions parametrized by this manifold. It is proved that solutions of Painlevé I equation have no poles at infinity at a given critical sector of the complex plane iff the related monodromy data belong to the singular submanifold. Such solutions coincide with the class of "truncated" solutions (intégrales tronquée) by classification of P. Boutroux. We derive further classification based on decomposition of singularities of monodromy data manifold.

Keywords: Painlevé equations · Tronquée solution
Distribution of poles · Riemann-hilbert problem
Complex manifold singularities · Padé approximations
Complex wkb method

1 Introduction

We study a distribution of poles of solutions to Painlevé I (PI)

$$u'' = 6u^2 - z, \tag{1}$$

It is well-known that all solutions to Painlevé equations of types I, II and IV are meromorphic in the complex plane [8]. Since the works of Paul Painlevé who proved this fact for Painlevé I equation, the distribution of poles became a matter of study in analytic theory of Painlevé transcendents. First studied by Boutroux in his papers [2] their asymptotics at infinity was proved to have regular lattice of poles, namely, any solution is approximated by a "deformed" elliptic function, with a modulus depending on $\frac{\arg z}{|z|}$. Indeed, the transformation of PI by $\zeta = \frac{4}{5}z^{5/4}$, $u = \sqrt{z}v$ yields

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$$v_{\zeta\zeta} - 6v^2 + 1 = \frac{4}{25}\frac{v}{\zeta^2} - \frac{v_\zeta}{\zeta}. \tag{2}$$

The right-hand side in (2) is small at infinity, which provides the deformation of the elliptic function v satisfying $v_{\zeta\zeta} - 6v^2 + 1 = 0$. Another feature established by Boutroux was a degeneration of the lattice of poles along the rays

$$\mathit{\Gamma}_n = \left\{ z \middle| \arg z = \frac{2\pi i}{5} n, \quad n = 0, 1, \ldots, 4 \right\}.$$

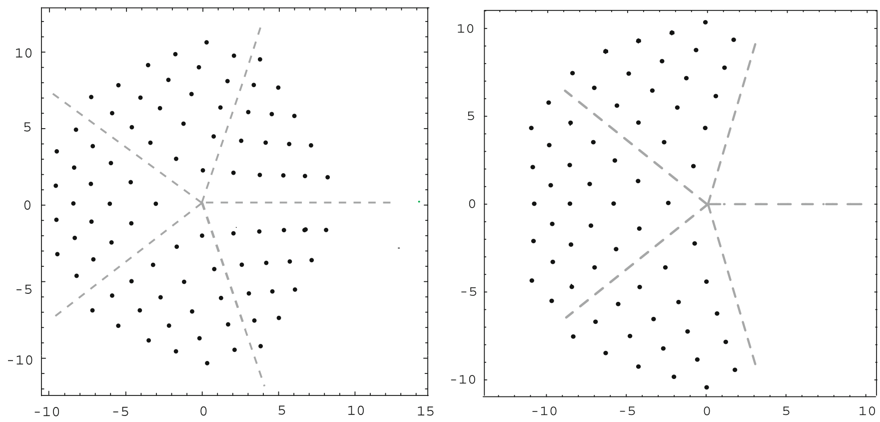

Fig. 1. Poles in the complex plane of generic (left) and 1-truncated (right) solutions of PI equation. Dashed lines denote the critical rays $\mathit{\Gamma}_n$.

These rays are called *critical*; degenerate lattice of poles comes from the solution

$$v(\zeta) = -1 + \frac{3}{\sin^2 \sqrt{3}(\zeta - \zeta_0)}$$

of the truncated Eq. (2) $v_{\zeta\zeta} - 6v^2 + 1 = 0$. Here imaginary period of elliptic function becomes infinite and poles go along the rays $\mathit{\Gamma}_0$, $\mathit{\Gamma}_2$ and $\mathit{\Gamma}_3$ in variable z. For non-degenerate elliptic solution, according to Boutroux [2], its poles are lying along *lines of poles* which are smooth curves tending asymptotically to the rays $\mathit{\Gamma}_n$ at infinity (see Fig. 1). Later in [6], Chap. 8 this was proved by establishing the "deformed" elliptic function asymptotics at infinity.

Boutroux proved also that for each ray there is a one-parameter family of particular solutions called truncated solutions (*intégrales tronquées*) whose lines of poles truncate for large z. He proved that the *intégrale tronquée* has no poles for large $|z|$ within two consecutive sectors bounded by the rays $\mathit{\Gamma}_{k-1}$, $\mathit{\Gamma}_k$ and $\mathit{\Gamma}_{k+1}$.

Nowadays, the structure of meromorphic solutions to Painlevé equations became more clear. On one hand, the method of isomonodromic deformations [6] gave an effective description of the poles distribution as $z \to \infty$. On the other hand, a numeric simulation gives a picture of poles in any compact domain of the complex plane. One of such numeric methods is based on Padé approximations. It was developed in [16] and proved to be good enough even for asymptotic distribution of poles at infinity. The pictures in Figs. 1 and 4 obtained by this method reveal that some well-known PI transcendents are the truncated solutions. We call a solution to be *n-truncated* if it has no poles along n critical rays Γ_k at infinity. For example, the Hastings-McLeod solution of Painlevé II equation shown in Fig. 3 is 2-truncated zero-parameter solution. It vanishes exponentially at positive half-axis and has square-root asymptotics at negative half-axis. Similarly, the *tritronquées* solution shown in Fig. 4 (left) is 3-truncated zero-parameter solution with square-root asymptotics in 4 of 5 sectors of the complex plane.

These experiments give a guideline for analytic description of truncated solutions. An analytic framework is provided by the method of isomonodromic deformations. It parametrizes all solutions in terms of coordinates on certain two-dimensional manifold. The coordinates are just integrals of motion which determine the unique solution of the Painlevé equation ([6], Chap. 5). The manifold is an algebraic one, it is defined by three polynomial equations in $\mathbb{C}^5$. Thus it has singularities where the global parametrization fails and some coordinates go infinite or zero in a particular chart. The submanifold of singularities is also an algebraic one, it has one- and zero-dimensional components. We prove [17] that submanifold of singularities parametrizes the set of truncated solutions.

The proof is based on asymptotic distribution of poles given by the complex WKB method applied for "undressing" of the Lax pair equations. The WKB method provides the leading terms as $|z| \to \infty$ for the poles in terms of monodromy data. The explicit formulas resemble the well-known quasi-classic quantization conditions of Bohr-Sommerfeld type for polynomial potential [15]. Thus the monodromy data are linked with asymptotic position of poles in the complex plane, so that vanishing conditions for poles along a certain ray Γ_n follow immediately. They form a singularity manifold which contains all truncated solutions and vice versa, since all derivations are explicit, no other solution belong to this submanifold.

Finally we discuss degenerations of the singularity manifold. One-dimensional components form 1-truncated solutions (with only one no-poles ray), while zero-dimensional (single points) relate to 2- or 3-truncated solutions. The latter was called by P. Boutrox as *intégrales tritronquée*. Unlike the case of Painlevé II equation, we prove that 2-truncated solutions are absent in PI equation. Also, we discuss the global behavior of poles of such zero-parameter solutions and recent proof the well-known Dubrovin's conjecture for 3-truncated solutions.

2 Poles Parametrization in General Case

Isomonodromic Method

For generic solution of PI equation we use isomonodromic deformation method described in [6, Chap. 5]. Below we mention briefly only main formulas and give asymptotic distributions of poles in the complex plane at infinity. Note that initially this approach was applied in [10, Chap. 10], where the poles asymptotics has been found on the real line as $|z| \to \infty$ in terms of the monodromy data. It was developed in [11,15] where the poles distributions were found for PII and PI equations in the sectors Ω_n of the complex plane.

Equation (1) is solved by linear matrix equation on function $\Psi = \Psi(\lambda, z)$ ([6], Chap. 5)

$$\Psi_\lambda = \begin{pmatrix} u_z & 2\lambda^2 + 2u\,\lambda - z + 2u^2 \\ 2(\lambda - u) & -u_z \end{pmatrix} \Psi. \tag{3}$$

Canonical solutions $\Psi_k(\lambda, z)$ of Eq. (3) are defined by asymptotics

$$\Psi_k(\lambda, z) \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda^{1/4} & \lambda^{1/4} \\ \lambda^{-1/4} & -\lambda^{-1/4} \end{pmatrix} \left[1 - \frac{1}{\sqrt{\lambda}} \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix} \right.$$
$$\left. + \frac{1}{2\lambda} \begin{pmatrix} H^2 & u \\ u & H^2 \end{pmatrix} + O\left(\lambda^{-3/2}\right) \right] e^{\theta(\lambda,z)\,\sigma_3}, \quad |\lambda| \to \infty, \quad \lambda \in \Sigma_k$$

which are valid in the sectors

$$\Sigma_k = \left\{ \lambda \in \mathbb{C} \,\middle|\, \frac{2\pi}{5}\left(k - \frac{3}{2}\right) < \arg\lambda < \frac{2\pi}{5}\left(k + \frac{1}{2}\right) \right\}, \quad k \in \mathbb{Z}.$$

Here we put

$$\theta(\lambda, z) = \frac{4}{5}\lambda^{5/2} - z\,\lambda^{1/2}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{2}\,u_z^2 - 2\,u^3 + z\,u,$$

and the cut for square root in the λ-plane is at the negative half-axis.

The Stokes matrices S_k of Eq. (3) have the form

$$\Psi_{k+1}(\lambda, z) = \Psi_k(\lambda, z)\, S_k, \quad \lambda \in \Sigma_k \cap \Sigma_{k+1}. \tag{4}$$

They are triangular

$$S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \quad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}$$

and satisfy cyclic relations

$$S_{k+5} = \sigma_1\, S_k\, \sigma_1, \qquad k \in \mathbb{Z}; \quad S_1 S_2 S_3 S_4 S_5 = i\,\sigma_1 \tag{5}$$

Monodromy data s_k are independent of z iff $u(z)$ satisfies equation PI (1). Due to relations (5) the monodromy manifold has the form [12]

$$\mathscr{P}_1 = \left\{ (s_1, s_2, s_3, s_4, s_5) \in \mathbb{C} \,\middle|\, s_{k+5} = s_k,\ s_{k+5} = i(1 + s_{k+2}s_{k+3}), \quad k = 0, \pm 1, \pm 2, \ldots \right\}.$$

Another form of two-dimensional complex manifold $\mathscr{P}_1$ is given by the vector potential function $\mathbf{F}$

$$\mathscr{P}_1 = \mathbb{C}^5/\{\mathbf{F} = 0\}, \quad \mathbf{F} = \begin{pmatrix} s_1 + s_3 + s_1 s_2 s_3 - i \\ s_2 + s_4 + s_2 s_3 s_4 - i \\ s_3 + s_5 + s_3 s_4 s_5 - i \end{pmatrix}. \tag{6}$$

Solving these relations with respect to s_2 and s_3 one can parametrize manifold $\mathscr{P}_1$ in the local chart

$$s_1 = \frac{i - s_3}{1 + s_2 s_3}, \quad s_4 = \frac{i - s_2}{1 + s_2 s_3}, \quad s_5 = i(1 + s_2 s_3) \quad \text{as} \quad 1 + s_2 s_3 \neq 0, \quad (7)$$

$$s_2 = s_3 = i, \quad s_5 = 0, \quad s_1 + s_4 = i, \quad \text{as} \quad 1 + s_2 s_3 = 0. \quad (8)$$

Generic Distribution of Poles

Applying complex WKB method [13] to linear equation (3) one finds asymptotic solutions of PI at the sectors Ω_n bounded by critical rays Γ_n and Γ_{n+1}

$$\Omega_n = \left\{ z \middle| \frac{2\pi i}{5} n < \arg z < \frac{2\pi i}{5}(n+1), \quad n = 1, 2, \ldots, 5 \right\}.$$

The results below were found initially in the papers [11,12,14]. They provide asymptotic distribution of poles for generic case, i.e. for the case of non-singular point at the monodromy manifold $\mathscr{P}_1$ (6).

Consider for determinacy the case $s_5 s_2 \neq 0$ and $z \in \Omega_5$. Then the point $(s_1, s_2, \ldots, s_5) \in \mathscr{P}_1$ is non-singular due to (6) and (7). Following [12,14], the asymptotics of PI solution has the form

$$u(z) = \sqrt{|z|}\wp\left(-\frac{4}{5}e^{i\varphi}|z|^{5/4} - \chi \middle| g_2, g_3\right)\left(1 + O(z^{-1})\right), \quad z \to \infty, \quad (9)$$

where $\wp$ is Weierstrass function with invariants

$$g_2 = 2e^{i\varphi}, \quad g_3 = A(\varphi), \quad (10)$$

and the phase shift is given by formula

$$\chi = \frac{1}{2\pi i}\left(\omega_1 \ln i s_2 + \omega_2 \ln \frac{s_5}{s_2}\right). \quad (11)$$

Here ω_1 and ω_2 are basic periods of Weierstrass function $\wp$

$$\omega_{1,2} = \int_{\mathscr{L}_1, \mathscr{L}_2} \frac{d\lambda}{w(\lambda)}. \quad (12)$$

They are determined on the Riemann surface

$$w^2 = \lambda^3 - \frac{1}{2}e^{i\varphi}\lambda + A(\varphi) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3),$$

and contours $\mathscr{L}_1$ and $\mathscr{L}_2$ are chosen in the form $\frac{1}{2}\mathscr{L}_1 = (\lambda_1, \lambda_2)$, $\frac{1}{2}\mathscr{L}_2 = (\lambda_3, \lambda_2)$ on the upper sheet of the surface.

Function $A = A(\varphi)$ in Eq. (10) of the Riemann surface is found as solution of integral equations

$$\mathrm{Re} \int_{\mathscr{L}_1, \mathscr{L}_2} w(\lambda) d\lambda = 0. \quad (13)$$

Together with symmetries and boundary conditions

$$A(0) = -\frac{2\sqrt{2}}{3\sqrt{3}}, \quad A(-\varphi) = \bar{A}(\varphi), \quad A\left(\varphi - \frac{2\pi}{5}\right) = A(\varphi)e^{2\pi i/5}. \qquad (14)$$

Equation (13) are known as *Boutroux problem*. It has unique solution $A = A(\varphi)$ ([12], p.245) which determines Weierstrass elliptic function with invariants (10).

The asymptotics (9) yields the following distribution of poles in the sector Ω_5

$$\frac{4}{5}e^{i\varphi}|z_{m,n}|^{5/4} + \chi = -\omega_1\, n - \omega_2\, m, \quad n, m \in \mathbb{Z}^+,$$

or, with respect to (11)

$$\tfrac{4}{5} z_{m,n}|z_{m,n}|^{1/4} = -\omega_1\left(\frac{\ln(is_2)}{2\pi i} + n\right) - \omega_2\left(\frac{\ln(s_2/s_5)}{2\pi i} + m\right) + o(1),$$
$$n, m \to +\infty, \quad n, m \in \mathbb{Z}^+, \quad z_{m,n} \in \Omega_5. \qquad (15)$$

Asymptotic formulas (15) clarify P. Boutroux's property of the lines of poles which tend to critical rays at infinity (see Introduction). Indeed, fixing one integer parameter, say $m = \text{const}$ and $n \to +\infty$, we have the sequence of $z_{m,n}$ lying on a smooth line which go to infinity. The distance between this line and a critical ray is not necessarily tend to zero, in fact, most lines of poles come parallel to critical rays at infinity. Thus the definition of the Boutroux property should be as follows.

Definition 2.1. Let Γ_n^ε be the rays close to Γ_n:

$$\Gamma_n^\varepsilon = \left\{ z \middle| \arg z = \frac{2\pi i}{5} n + \varepsilon, \quad n = 1, 2, \ldots, 5, \quad \varepsilon \to 0 \right\}.$$

We say that a line of poles tends to the ray Γ_n in the sector Ω_n if it crosses rays Γ_n^ε for any $\varepsilon > 0$.

3 Singularities of the Monodromy Manifold $\mathscr{P}_1$

We define the singular points of the monodromy manifold as those having at least one coordinate s_j to be zero or infinity, i.e. $\tilde{\mathscr{P}}_1 \subset \mathscr{P}_1$ is singular submanifold if

$$\tilde{\mathscr{P}}_1 = \left\{ \mathbf{s} \in \mathscr{P}_1 \middle| \{s_j = 0\} \cup \{s_j = \infty\},\ \mathbf{s} = (s_1, \ldots, s_j, \ldots) \right\}.$$

In terms of potential function $\mathbf{F}$ this submanifold reads

$$\tilde{\mathscr{P}}_1 = \left\{ \mathbf{s} \in \mathscr{P}_1 \middle| \mathbf{F}_{s_j} = 0, \quad j = 1, 2, \ldots, 5 \right\}. \qquad (16)$$

In the local map (7) it is given explicitly

$$\tilde{\mathscr{P}}_1 = \{\{s_2 = s_3 = i\} \cup \{s_5 = 0\} \cup \{s_1 + s_4 = i\}\}. \qquad (17)$$

Theorem 3.1. *A solution of PI is truncated iff it is parametrized by the point from singular submanifold (16).*

Proof. Prove first the necessity. Without loss of generality, consider truncated solutions which have no line of poles in Ω_5 tending to Γ_0. Else one can use symmetries

$$u(z) \mapsto e^{-\frac{4\pi i}{5}} u\left(e^{\frac{2\pi i}{5}} z\right) \tag{18}$$

to consider other critical rays.

Assume, on the contrary, that a truncated solution $u(z)$ have the coordinates $\mathbf{s} \in \mathscr{P}_1$ but $\mathbf{s} \in \tilde{\mathscr{P}}_1$ so that $s_5 \neq 0$ and $s_5 \neq 0$. Then solution $u(z)$ has asymptotics (9) in the sector Ω_5 with distribution of poles given by (15). Consider this distribution at the ray Γ_0^ε, i.e. in the limit $\arg z = \varepsilon \to 0$.

In this limit, due to (14) we have $w^2 = \lambda^3 - \frac{1}{2}\lambda - (2/3)^{3/2}$ and complete elliptic integrals for periods (12) are calculated explicitly

$$\omega_1 \to ia, \quad \omega_2 \to -b - ic,$$

where $a = 0.265984...$, $b = 0.756164...$, $c = 0.336546....$ Then asymptotic expansion (15) yields

$$\tfrac{4}{5} z_{m,n} |z_{m,n}|^{1/4} = i(cm - an) + bm + \frac{a\ln(is_2) + (c+b)\ln(s_2/s_5)}{2\pi} + o(1),$$
$$n, m \to +\infty, \quad n, m \in \mathbb{Z}^+, \quad \varepsilon \to 0.$$

Choose the integers m, n by condition

$$cm - an = O(1), \quad n, m \to +\infty,$$

then the line of poles with points $z_{m,n}$ has the argument $\arg z_{m,n} = O(m^{-1}) < \varepsilon$ for any $\varepsilon > 0$ and $m > \varepsilon^{-1}$. The contradiction proves the necessity statement of the theorem.

The sufficiency statement follows from the results of papers [11,12]. With our notations we formulate them as the following lemma.

Lemma 3.2. **(**[12], **Theorems 3, 4 and Corollary)** *If $s_5 = 0$, then the solution of PI equation has the regular asymptotic expansion in the sector $\Omega_4 \cup \Omega_5$*

$$u(z) = \sqrt{\frac{z}{6}} \left(1 + \frac{\theta(\varphi)s_4 - \theta(\varphi)s_2}{\sqrt{10\pi t}} e^{-2t} + O(|z|^{-2})\right), \quad z \to \infty, \quad -\frac{2\pi}{5} < \arg z < \frac{2\pi}{5}, \tag{19}$$

where

$$t = \frac{8}{5}\sqrt[4]{\frac{3}{2}} z^{5/4}, \quad \theta(\varphi) = \begin{cases} 0, \ \varphi < 0, \\ \frac{1}{2}, \ \varphi = 0, \\ 1, \ \varphi > 0, \end{cases} \quad \varphi = \arg z. \tag{20}$$

Lemma 3.2 covers a part of submanifold $\tilde{\mathscr{P}}_1$ by choosing 1-truncated solutions which have no poles along the critical ray Γ_0. Other parts of $\tilde{\mathscr{P}}_1$ can be considered with the help of discrete symmetry (18). The correspondence between equations $s_j = 0$ and truncation along the critical rays is shown in diagram Fig. 2, right (see Theorem 4.1 below).

4 Truncated Solutions of PI Equation

Here we present examples of all types of truncated solutions related to singularities of the monodromy manifold. They are based on results by Kapaev and Kitaev in the papers [11, 12].

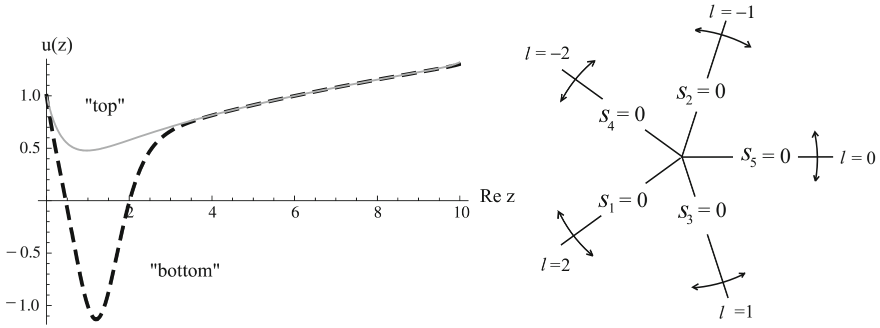

Fig. 2. Two 1-truncated solutions with $s_1+s_4 = i$ (left) and diagram for correspondence between relations $s_{5-2l} = 0$ and critical rays where the solution truncates [12] (right) The arrows mark adjacent sectors where regular asymptotics hold.

4.1 1-Truncated Solutions

Let $\mathbf{s} \in \tilde{\mathscr{P}}_1$ and there is one coordinate $s_j \neq 0, i$, i.e. $\tilde{\mathscr{P}}_1$ is not degenerate at the point $\mathbf{s}$. In this case, the regular asymptotics of the 1-truncated solution is described by the following theorem

Theorem 4.1. **(***[12],* **Theorem 3)** *Let $s_{5-2l} = 0$, $l = 0, \pm 1, \pm 2$ and $u(z)$ be the solution of PI equation with such monodromy data. Then this solution is 1-truncated along the critical ray Γ_{4-l} and have regular asymptotics*

$$u(z) = \sqrt{\frac{z}{6}}\left(1 + O(z^{-2})\right), \quad z \to \infty, \quad z \in \Omega_{4-l} \cup \Omega_{5-l}. \tag{21}$$

The diagram of all cases listed in Theorem 4.1 is shown in Fig. 2 right. For $l = 0$ the asymptotics as $-2\pi/5 < \arg z < 2\pi/5$ is given by Lemma 3.2. Two of these 1-truncated solutions on positive half-axis are shown in Fig. 2. Following [3] they are called "top" and "bottom" solutions with $u(0) = 1$, $u'(0) = -1.818275\ldots$ and with $u(0) = 1$, $u'(0) = -2.765598\ldots$ respectively. In Sect. 5 we investigate dynamics of poles of "intermediate" solutions while deforming one of them into another by changing the derivative $u'(0)$ between those values.

4.2 2-Truncated Solutions

These solutions should have no poles along two critical rays. They correspond to zero-parameter degenerations of submanifold $\tilde{\mathscr{P}}_1$. However, unlike Painlevé II equation (PII)

$$u'' = zu + 2u^3 \tag{22}$$

there are no such solutions in PI equation.

Recall briefly that 2-truncated solution of PII Eq. (22) is known as Hastings-McLeod one [9]. It is a zero-parameter solution corresponding to singular points of PII monodromy data manifold $\mathscr{P}_2$ [17]

$$s_1 = \pm i, \quad s_2 = 0, \quad s_3 = \mp i,$$

where $\mathscr{P}_2 = \{(s_1, s_2, s_3) \in \mathbb{C} \,|\, s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0\}$. On the real axis it has regular asymptotics [6, 10] (see Fig. 3 left)

$$\begin{aligned} u(z) &= \mp \frac{1}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) + O(z^{-3/4}), \quad z \to +\infty, \\ u(z) &= \pm \sqrt{-\frac{z}{2}} \left(1 \mp \frac{1}{9z^3} + \frac{73}{128 z^6} + O(|z|^{-9})\right), \quad z \to -\infty. \end{aligned} \tag{23}$$

There are six critical rays $\varGamma_n$ for solutions of PII equation (22), and Hastings-McLeod solution is free of poles along two rays (see Fig. 3 right). Numeric experiments show that *all* poles of this solution are located in the sectors $\pi/3 < \arg z < 2\pi/3$ and $4\pi/3 < \arg z < 5\pi/3$ bounded by the critical rays (see Fig. 3). Thus the following proposition should be true

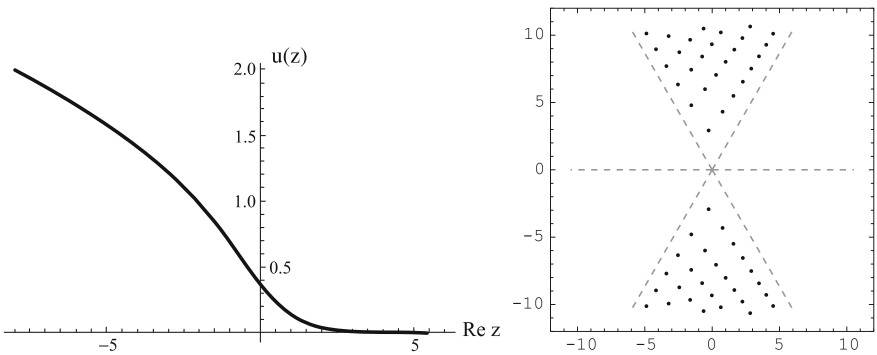

Fig. 3. Hastings-McLeod 2-truncated solution (left) and its distribution of poles (right) [16].

Conjecture 4.2. The Hastings-McLeod solution (23) of equation PII has no poles in the sectors $|\arg z| < \pi/3$ and $2\pi/3 < \arg z < 4\pi/3$.

Recently, the "half" of the conjecture was proved in the paper [1] by Bertola. Namely, he proved that Hastings-McLeod solution has no poles in the sector $|\arg z| < \pi/3$. The proof is based on special representation of the solution in terms of a Fredholm determinant which came from isomonodromic nature of PII equation. An accurate estimate of the kernel of the Fredholm operator shows that it is positive in the given sector, thus the Fredholm determinant has no zeroes (they coincide with poles of the PII solution). Unfortunately, this estimate does not work for another sector $2\pi/3 < \arg z < 4\pi/3$.

Finishing the digression on 2-truncated solutions note that conjectures on their poles distributions are important because a number of applications. One of them comes from the theory of random matrices and combinatorics of permutations where the Hastings-McLeod solution enters into distribution functions of central limit theorems for Gaussian unitary ensembles of $n \times n$ matrices as $n \to \infty$ [18].

Returning to PI equation note a possible reason of absence of 2-truncated solutions. Since PI equation has 5-fold symmetry (18) instead of 6-fold symmetry for PII equation, there are less possibilities for truncated solutions in more than one sector. Indeed, the structure of monodromy manifold defined by equations $\mathbf{F} = 0$ (16) provides the following "no 2-truncated" theorem.

Theorem 4.3. *There are only 1- and 3-truncated solutions of PI equation.*

Proof. Suppose there is a solution $u(z)$ which have regular asymptotics at infinity along two critical rays. Then, by Theorem 4.1 it has two monodromy coordinates vanish, say $s_j = 0$ and $s_k = 0$. If $j = k - 1$ or $j = k + 1$ then from the diagram Fig. 2 it is clear that $u(z)$ has regular asymptotics (21) in four sectors Ω_n. Thus $u(z)$ has no poles along three critical rays, i.e. it is 3-truncated solution. The possibility $j = k - 2$ or $j = k + 2$ is impossible because equation $\mathbf{F} = 0$ does not hold for $s_{k-2} = 0$ and $s_k = 0$ or $s_{k+2} = 0$ and $s_k = 0$. The contradiction proves the theorem.

3-Truncated Solutions
These solutions correspond to the "boundary edges" of the submanifold $\tilde{\mathscr{P}}_1 \subset \mathscr{P}_1$, i.e. to the points $s_k = 0$, $s_k + s_{k+3} = i$, $k = 1, 2, \ldots, 5$. In fact, this is only one solution (with respect to discrete symmetries (18)), given by coordinates

$$s_2 = s_3 = s_4 = i, \quad s_1 = s_5 = 0.$$

The distribution of its poles is shown in Fig. 4 left. On the negative half-line it is singular and has infinite number of poles. On the positive half-line and in the sectors Ω_4, Ω_5, Ω_1 and Ω_2 it has regular asymptotics

$$u(z) = -\sqrt{\frac{z}{6}} - \frac{1}{48\,z^2} - \frac{49}{768\,z^5}\sqrt{\frac{z}{6}} - \frac{1225}{9216\,z^7} - \frac{4412401}{1179648\,z^{10}}\sqrt{\frac{z}{6}} + O\left(z^{-12}\right), \quad z \to \infty, \tag{24}$$

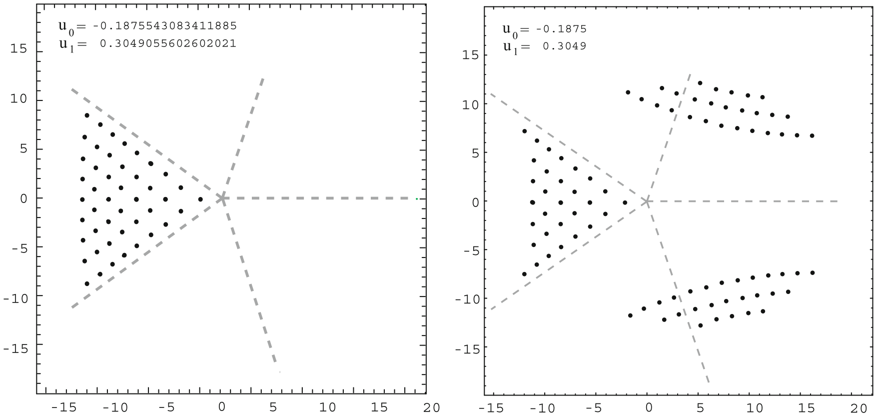

Fig. 4. 3-truncated (tritronquée) solution (left) and its approximation (right). The initial data $u_0 = u(0)$ and $u_1 = u'(0)$ is shown in upper left corner.

which is similar to (19) with no exponential terms since $s_2 = s_4$.

Note that asymptotic expansion (24) is valid at infinity and does not garantee the absence of poles everywhere in these four sectors. The lines of poles near critical rays Γ_3 and Γ_4 might cross them in a bounded domain and put some poles into Ω_2 or Ω_4. This phenomenon can be seen for 1-truncated "heaven" solution shown in Fig. 5, lower right. Nevertheless, numeric experiments done in the papers [5, 7, 16] show that all poles are located in sector Ω_2. The following proposition is true

Proposition 4.4. *3-truncated solution of equation PI given by asymptotics (24) has no poles in the sector* $|\arg z| < 4\pi/5$.

This proposition was known for a long time as Dubrovin's conjecture. It first appeared in [5] where the authors studied a wave collapse in nonlinear Schrödinger equation. They derived a solution of PI as the leading term of asymptotics at the moment of gradient catastrophe. This asymptotics should be non-singular in half-plane $\Re z > 0$ which required the PI solution to be free of poles everywhere except the critical sector Ω_2.

Recently the proof of Dubrovin's conjecture was given in paper [4]. The idea is to make an analytic continuation of solution represented by the series (24) into a bounded domain near the origin. This continuation shows that 3-truncated solution is analytic everywhere except the sector Ω_2 and the circle $|z| < 1.7$. Then a number of sophisticated estimates are applied to prove analyticity in this circle. It can be seen that analyticity in the sector is true for all solutions close to the 3-truncated because the first pole of this solution is $z_{0,0} = -2.384168\ldots$.

The proof does not apply the integrability property of PI equation. However, Theorems 3.1 and 4.3 show that monodromy data manifold is essential for distribution of poles of corresponding solution. The use of this correspondence can clarify analytic properties of other special solutions, especially for higher Painlevé equations.

5 Formation of Truncated Solutions

While doing Padé approximation of truncated solutions one can see poles of two types: the part that is almost still and lie in positions of that of truncated solution and an external part which disappear with better approaching to the truncated solution (Fig. 4 right). This effect is inevitable in numerics since the initial or monodromy data always come with some error. Suppose that one knows the initial data $u(0)$ and $u'(0)$ of a truncated solution with a precision $\varepsilon \ll 1$

$$u(0) - u_{approx}(0) = \varepsilon, \quad u'(0) - u'_{approx}(0) = \varepsilon.$$

The poles of the external part move with a speed $O(\varepsilon^{-4/5})$ towards infinity in a transverse directions to the line of poles. This can be seen from asymptotic distribution of poles (9) and (11). Indeed, the approximate solution u_{approx} for 1-truncated solution with $s_5 = 0$ has monodromy data close to the truncated one

$$s_2^{approx} = s_2 + O(\varepsilon), \quad s_5^{approx} = O(\varepsilon).$$

Thus the phase shift (11) goes to infinity

$$\chi^{approx} = O(\ln \varepsilon), \quad \varepsilon \to 0,$$

because the periods ω_1 and ω_2 do not depend on ε. The asymptotics of poles (15) yields $|z_{m,n}| = O(\chi^{4/5})$, so that the speed of external poles moving to infinity is

$$\frac{\partial |z_{m,n}|}{\partial \varepsilon} = O(\varepsilon^{-4/5}), \quad \varepsilon \to 0.$$

This gives a receipt to check the calculations of the poles positions. While running the numeric procedure a number of times with various precision of initial data one can distinguish the true poles among others that move fast from initial positions. In fact, it is a practical application of the basic property of "movable poles" which comes form the very definition of Painlevé equations.

The similar picture is seen while doing deformation of one truncated solution into another. The example of two 1-truncated solutions mentioned in Sect. 4.1 is an illustration of non-trivial structure of their poles distributions.

Consider two solutions of PI equations with regular asymptotics (21) at the sectors $\Omega_4 \cup \Omega_5$ with the following initial data [3]

$$u_{top}(0) = 1, \quad u'_{top}(0) = -1.818275\ldots$$
$$u_{bottom}(0) = 1, \quad u'_{bottom}(0) = -2.765598\ldots,$$

Their monodromy data belong to the singular submanifold $\tilde{\mathscr{P}}_1$ with $s_5 = 0$. On the positive half-line these solutions are shown in Fig. 2 left. Let us use the derivative $u'(0)$ as the deformation parameter and view the poles of the "intermediate" solutions. Some pictures of this deformation are shown in Fig. 5.

The "top" solution has no poles in two right sectors Ω_4 and Ω_5. Then some lines of poles come from the right infinity and approach critical rays Γ_1 and

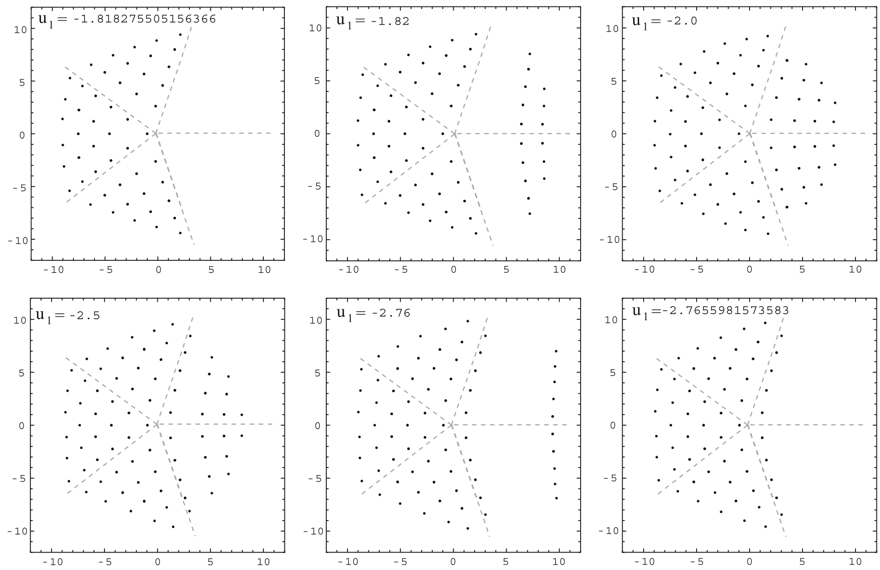

Fig. 5. Evolution of poles while tranforming "bottom" solution (upper left) into "top" solution (lower right). Initial values are $u_0 = u(0) = 1$ and $u_1 = u'(0)$ is changing as shown in each figure.

$\varGamma_4$ (three upper pictures in Fig. 5). Further deformation leads to some moment when the lines of poles start to move back. However, one of them closest to the rays $\varGamma_1$ and $\varGamma_4$ "glues" to these rays and does not move any more (three lower pictures). This line of poles passes through the sectors $\varOmega_4$ and $\varOmega_5$, although at infinity it is out of these sectors.

The similar scenario occurs an infinite number of times while changing $u_n \to -\infty$, where $u'(0) = u_n$, $u(0) = 1$ correspond to a family of 1-truncated solutions on the positive half-line.

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Elliptic Calogero-Moser Hamiltonians and Compatible Poisson Brackets

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Abstract. We show that the classical elliptic A_2-Calogero-Moser Hamiltonian is generated by the elliptic quadratic Poisson bracket of the $q_{9,2}(\tau)$-type.

Keywords: Calogero-Moser Hamiltonians · Elliptic Calogero-Moser Hamiltonians · Poisson brackets

1 Introduction

It was shown in [3] that there exist commutative subalgebras in the universal enveloping algebras $U(gl_3)$ and $U(gl_4)$ such that some their representations by differential operators give rise to elliptic quantum Calogero-Moser Hamiltonians [1,4] with three and four particles respectively.

A class of commutative subalgebras in $U(gl_3)$ was constructed in [8]. These subalgebras are quantizations of commutative Poisson subalgebras generated by compatible constant and linear $\mathfrak{gl}_n$-Poisson brackets. The quantization recipe is very simple: any product $\prod_1^k x_i$ of commuting generators should be replaced by $\frac{1}{k!}\sum_{\sigma\in S_k}\prod y_{\sigma(i)}$, where y_i are non-commutative generators.

We discover that the commutative subalgebra in $U(gl_3)$ found in [3] is a quantization of the same type for a Poisson subalgebra related to compatible quadratic elliptic [2] and linear $\mathfrak{gl}_3$-Poisson brackets.

2 Polynomial Forms for Elliptic Quantum Calogero-Moser Hamiltonians

In this Section, we recall the results of [3] for the sake of completeness. The differential operator

$$H_N = -\Delta + \beta(\beta-1)\sum_{i\neq j}^{N+1}\wp(x_i - x_j), \tag{1}$$

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is called quantum elliptic Calogero-Moser Hamiltonian. Here, $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$, β is a parameter, and $\wp(x)$ is the Weierstrass $\wp$-function with the invariants $g_2,\ g_3$. In the coordinates

$$X = \frac{1}{N+1}\sum_{i=1}^{N+1} x_i, \qquad y_i = x_i - X$$

operator (1) takes the form

$$H_N = -\frac{1}{N+1}\frac{\partial^2}{\partial X^2} + \mathscr{H}_N\left(y_1, y_2, \dots y_N\right),$$

where

$$\mathscr{H}_N = -\frac{N}{N+1}\sum_{i=1}^{N}\frac{\partial^2}{\partial y_i^2} + \frac{1}{N+1}\sum_{i\neq j}^{N}\frac{\partial^2}{\partial y_i \partial y_j} + \beta(\beta-1)\sum_{i\neq j}^{N+1}\wp(y_i - y_j). \tag{2}$$

In the last term we have to substitute $y_{N+1} = -\sum_{i=1}^{N} y_i$.

In [3] the following transformation $(y_1, \dots, y_N) \to (u_1, \dots, u_N)$ defined by

$$\begin{pmatrix} \wp(y_1) & \wp'(y_1) & \dots & \wp^{(N-2)}(y_1) & \wp^{(N-1)}(y_1) \\ \wp(y_2) & \wp'(y_2) & \dots & \wp^{(N-2)}(y_2) & \wp^{(N-1)}(y_2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \wp(y_N) & \wp'(y_N) & \dots & \wp^{(N-2)}(y_N) & \wp^{(N-1)}(y_N) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{3}$$

was considered. Denote by $D_N(y_1, \dots, y_N)$ the Jacobian of transformation (3).

Conjecture 1. The gauge transformation $\mathscr{H}_N \to D_N^{-\frac{\beta}{2}}\mathscr{H}_N D_N^{\frac{\beta}{2}}$ and subsequent change of variables (3) bring (2) to a differential operator P_N with polynomial coefficients.

In the case $N = 2$, transformation (3) coincides with the transformation

$$u_1 = \frac{\wp'(y_2) - \wp'(y_1)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}, \qquad u_2 = \frac{\wp(y_1) - \wp(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}, \tag{4}$$

found in [7]. For rational and trigonometric degenerations see [5].

It is easy to verify that the operators $e_{ij} = E_{i-1,j-1}$, where

$$\begin{gathered} E_{ij} = y_i\frac{\partial}{\partial y_j}, \qquad E_{0i} = \frac{\partial}{\partial y_i}, \\ E_{00} = -\sum_{j=1}^{n-1} y_j\frac{\partial}{\partial y_j} + \beta\, n, \qquad E_{i0} = y_i E_{00} \end{gathered} \tag{5}$$

satisfy the commutator relations

$$e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{j,k}e_{il} - \delta_{i,l}e_{kj}, \qquad i, j = 1, \dots, N+1, \tag{6}$$

and, therefore, define a representation of the Lie algebra gl_{N+1} and of the universal enveloping algebra $U(gl_{N+1})$. The latter representation is not exact.

Conjecture 2. The differential operators P_N can be written as a linear combinations of anti-commutators of the operators E_{ij}.

The conjectures 1 and 2 have been verified in [3] for $N = 2, 3$. Moreover, differential operators with polynomial coefficients that commute with P_N were found. These operators also can be written as non-commutative polynomials in E_{ij}. The representation of the above operators through E_{ij} is not unique. It turns out that for $N = 2, 3$ there exist such a representation that the corresponding non-commutative polynomials in e_{ij} commute with P_N as elements of the universal enveloping algebra $U(gl_{N+1})$.

Consider the case $N = 2$. It turns out that the element

$$H = H_0 + H_1 g_2 + H_2 g_2^2 + H_3 g_3, \tag{7}$$

where

$$H_0 = 12e_{12}e_{11} - 12e_{32}e_{13} - 12e_{33}e_{12} - e_{23}^2,$$
$$H_1 = -e_{21} + 2e_{21}e_{11} - e_{22}e_{21} - e_{31}e_{23} - 12e_{32}^2 - e_{33}e_{21},$$
$$H_2 = -e_{31}^2, \qquad H_3 = 36e_{32}e_{31} + 3e_{21}^2,$$

of the universal enveloping algebra $U(gl_3)$ defined by relations (6) commutes with two third order elements of the form

$$K = K_0 + K_1 g_2 + K_2 g_3,$$
$$M = M_0 + M_1 g_2 + M_2 g_3 + M_3 g_2^2 + M_4 g_2 g_3 + M_5 g_3^2 + M_6 g_2^3.$$

Here, g_2 and g_3 are invariants of the Weierstrass $\wp$-function from (1) and

$$K_0 = -e_{23} + 2e_{21}e_{13} - e_{23}e_{22} - 36e_{32}e_{12} + e_{33}e_{23} - e_{21}e_{13}e_{11} - e_{22}e_{21}e_{13} + e_{23}e_{11}^2 + 2e_{23}e_{21}e_{12} - e_{23}e_{22}e_{11} + 12e_{31}e_{12}^2 - e_{31}e_{23}e_{13} - 12e_{32}e_{12}e_{11} - e_{32}e_{23}^2 - 12e_{32}^2e_{13} + 2e_{33}e_{21}e_{13} - e_{33}e_{23}e_{11} + e_{33}e_{23}e_{22} + 12e_{33}e_{32}e_{12},$$

$$K_1 = 3e_{31}e_{11} - 3e_{31}e_{22} - 2e_{32}e_{21} + e_{31}e_{21}e_{12} + e_{31}e_{22}e_{11} - e_{31}e_{22}^2 + e_{31}^2e_{13} - 2e_{32}e_{21}e_{11} + e_{32}e_{22}e_{21} - 2e_{32}e_{31}e_{23} - e_{33}e_{31}e_{11} + e_{33}e_{31}e_{22} + e_{33}e_{32}e_{21},$$

$$K_2 = 3\left(2e_{31}e_{21} + e_{31}e_{22}e_{21} + e_{31}^2e_{23} - e_{32}e_{21}^2 - e_{33}e_{31}e_{21}\right);$$

$$M_0 = 2\Big(12e_{13}e_{11} - 6e_{22}e_{13} - 6e_{33}e_{13} - 12e_{13}e_{11}^2 - 6e_{22}e_{13}e_{11} + 6e_{22}^2e_{13} + 18e_{23}e_{12}e_{11} - 18e_{23}e_{22}e_{12} + e_{23}^3 - 216e_{32}e_{12}^2 + 18e_{32}e_{23}e_{13} + 30e_{33}e_{13}e_{11} - 6e_{33}e_{22}e_{13} - 12e_{33}^2e_{13}\Big),$$

$$M_1 = -3\Big(2e_{23}e_{21} - 36e_{31}e_{12} + 20e_{32}e_{11} - 28e_{32}e_{22} + 8e_{33}e_{32} - 4e_{23}e_{21}e_{11} + 2e_{23}e_{22}e_{21} - 12e_{31}e_{12}e_{11} - e_{31}e_{23}^2 + 8e_{32}e_{11}^2 + 36e_{32}e_{21}e_{12} + 4e_{32}e_{22}e_{11} - 4e_{32}e_{22}^2 - 24e_{32}e_{31}e_{13} - 12e_{32}^2e_{23} + 2e_{33}e_{23}e_{21} + 12e_{33}e_{31}e_{12} - 20e_{33}e_{32}e_{11} + 4e_{33}e_{32}e_{22} + 8e_{33}^2e_{32}\Big),$$

$$M_2 = -18\Big(4e_{31}e_{11} - 2e_{31}e_{22} - 2e_{33}e_{31} - e_{23}e_{21}^2 - 2e_{31}e_{11}^2 - 6e_{31}e_{21}e_{12} + 2e_{31}e_{22}e_{11} - 2e_{31}e_{22}^2 + 6e_{31}^2e_{13} + 6e_{32}e_{22}e_{21} + 24e_{32}^3 + 2e_{33}e_{31}e_{11} + 2e_{33}e_{31}e_{22} - 6e_{33}e_{32}e_{21} - 2e_{33}^2e_{31}\Big),$$

$$M_3 = -3\Big(2e_{31}e_{21} - 2e_{31}e_{21}e_{11} + e_{31}e_{22}e_{21} + e_{31}^2e_{23} - 24e_{32}^2e_{31} + e_{33}e_{31}e_{21}\Big),$$

$$M_4 = 9\Big(e_{31}e_{21}^2 - 12e_{32}e_{31}^2\Big), \qquad M_5 = 108e_{31}^3, \qquad M_6 = -2e_{31}^3.$$

One can verify that $[K, M] = 0$. Thus, we obtain a commutative subalgebra in $U(gl_3)$ generated by the elements H, K, M and by the three Casismir elements of $U(gl_3)$ of order 1, 2, and 3.

Different representations of $U(gl_3)$ by differential, difference and q-difference operators generate "integrable" operators. In particular, the representation by differential operators (5) with two independent variables maps the element H to a polynomial form P_2 for A_2-Calogero-Moser Hamiltonian, the element M to a third order differential operator that commutes with P_2, and K to zero.

Notice that the representation of $U(gl_3)$ by the matrix unities in Mat_3 maps H, K and M to zero.

The representation defined by

$$e_{ij} \to z_i \frac{\partial}{\partial z_j}$$

maps H to a homogeneous differential operator with 3 independent variables of the form $\mathscr{H} = \sum_{i \geq j} a_{ij} \frac{\partial^2}{\partial z_i \partial z_j}$, where

$$a_{11} = -2g_2 z_1 z_2 - 3g_3 z_2^2 + g_2^2 z_3^2, \qquad a_{22} = 12 g_2 z_3^2,$$
$$a_{33} = z_2^2, \qquad a_{21} = -12 z_1^2 + g_2 z_2^2 - 36 g_3 z_3^2,$$
$$a_{31} = 2\, g_2 z_2 z_3, \qquad a_{32} = 24\, z_1 z_3.$$

and M to an operator $\mathscr{M} = \sum_{i \geq j \geq k} b_{ijk} \frac{\partial^3}{\partial z_i \partial z_j \partial z_k}$. It is interesting that in both $\mathscr{H}$ and in $\mathscr{M}$ the lower order terms are absent.

3 Classical Case

Consider the following limit procedure. Any element $f \in U(gl_n)$ is a polynomial of the non-commutative variables e_{ij}, which obey the commutator relation (6). Taking all terms of the highest degree in f and replacing e_{ij} by commutative variables x_{ij} there, we obtain a polynomial $sym(f)$.

It is known that for any elements f and g of $U(gl_n)$

$$sym([f,\, g]) = \{sym(f),\, sym(g)\},$$

where $\{,\}$ is the linear Poisson bracket

$$\{x_{ij}, x_{kl}\} = \delta_{j,k}\, x_{il} - \delta_{i,l}\, x_{kj}, \qquad i, j = 1, \ldots, n, \tag{1}$$

which corresponds to the Lie algebra gl_n. In particular, if $[f, g] = 0$, then $\{sym(f),\, sym(g)\} = 0$.

So, now, we consider commutative polynomials in variables x_{ij}. We will regard them as the entries of a matrix X. Applying the limit procedure to the generators of commutative subalgebra in $U(gl_3)$ described in Sect. 2, we get the polynomials

$$\begin{gathered} c_1 = \operatorname{tr} X, \qquad c_2 = \operatorname{tr} X^2, \qquad c_3 = \operatorname{tr} X^3, \\ h = h_0 + h_1 g_2 + h_2 g_2^2 + h_3 g_3, \qquad k = k_0 + k_1 g_2 + k_2 g_3, \\ m = m_0 + m_1 g_2 + m_2 g_3 + m_3 g_2^2 + m_4 g_2 g_3 + m_5 g_3^2 + m_6 g_2^3, \end{gathered} \tag{2}$$

where

$$h_0 = 12x_{12}x_{11} - 12x_{32}x_{13} - 12x_{33}x_{12} - x_{23}^2,$$
$$h_1 = 2x_{21}x_{11} - x_{22}x_{21} - x_{31}x_{23} - 12x_{32}^2 - x_{33}x_{21},$$
$$h_2 = -x_{31}^2, \qquad h_3 = 36x_{32}x_{31} + 3x_{21}^2,$$
$$k_0 = -x_{21}x_{13}x_{11} - x_{22}x_{21}x_{13} + x_{23}x_{11}^2 + 2x_{23}x_{21}x_{12} - x_{23}x_{22}x_{11} + 12x_{31}x_{12}^2 - x_{31}x_{23}x_{13} - 12x_{32}x_{12}x_{11} - x_{32}x_{23}^2 - 12x_{32}^2x_{13} + 2x_{33}x_{21}x_{13} - x_{33}x_{23}x_{11} + x_{33}x_{23}x_{22} + 12x_{33}x_{32}x_{12},$$
$$k_1 = x_{31}x_{21}x_{12} + x_{31}x_{22}x_{11} - x_{31}x_{22}^2 + x_{31}^2x_{13} - 2x_{32}x_{21}x_{11} + x_{32}x_{22}x_{21} - 2x_{32}x_{31}x_{23} - x_{33}x_{31}x_{11} + x_{33}x_{31}x_{22} + x_{33}x_{32}x_{21},$$
$$k_2 = 3\,(x_{31}x_{22}x_{21} + x_{31}^2x_{23} - x_{32}x_{21}^2 - x_{33}x_{31}x_{21});$$
$$m_0 = 2\Big(- 12x_{13}x_{11}^2 - 6x_{22}x_{13}x_{11} + 6x_{22}^2x_{13} + 18x_{23}x_{12}x_{11} - 18x_{23}x_{22}x_{12} + x_{23}^3 - 216x_{32}x_{12}^2 + 18x_{32}x_{23}x_{13} + 30x_{33}x_{13}x_{11} - 6x_{33}x_{22}x_{13} - 12x_{33}^2x_{13}\Big),$$
$$m_1 = -3\Big(- 4x_{23}x_{21}x_{11} + 2x_{23}x_{22}x_{21} - 12x_{31}x_{12}x_{11} - x_{31}x_{23}^2 + 8x_{32}x_{11}^2 + 36x_{32}x_{21}x_{12} + 4x_{32}x_{22}x_{11} - 4x_{32}x_{22}^2 - 24x_{32}x_{31}x_{13} - 12x_{32}^2x_{23} + 2x_{33}x_{23}x_{21} + 12x_{33}x_{31}x_{12} - 20x_{33}x_{32}x_{11} + 4x_{33}x_{32}x_{22} + 8x_{33}^2x_{32}\Big),$$
$$m_2 = -18\Big(- x_{23}x_{21}^2 - 2x_{31}x_{11}^2 - 6x_{31}x_{21}x_{12} + 2x_{31}x_{22}x_{11} - 2x_{31}x_{22}^2 + 6x_{31}^2x_{13} + 6x_{32}x_{22}x_{21} + 24x_{32}^3 + 2x_{33}x_{31}x_{11} + 2x_{33}x_{31}x_{22} - 6x_{33}x_{32}x_{21} - 2x_{33}^2x_{31}\Big),$$
$$m_3 = -3\Big(- 2x_{31}x_{21}x_{11} + x_{31}x_{22}x_{21} + x_{31}^2x_{23} - 24x_{32}^2x_{31} + x_{33}x_{31}x_{21}\Big),$$
$$m_4 = 9\Big(x_{31}x_{21}^2 - 12x_{32}x_{31}^2\Big), \qquad m_5 = 108x_{31}^3, \qquad m_6 = -2x_{31}^3.$$

These polynomials commute with each other with respect to the linear $\mathfrak{gl}_3$-Poisson bracket (1).

It can be verified that the elements of the universal enveloping algebra can be reconstructed from polynomials (2) by the quantization procedure described in the introduction.

The following statement can be verified straightforwardly.

Proposition 1. Suppose a quadratic Poisson bracket $\{,\}$ is compatible with the bracket $\{f,g\}_1$ given by (1), where $n = 3$, and polynomials (2) commute with each other with respect to $\{,\}$. Then, $\{,\}$ is given by

$$\{f,g\} = \{f,g\}_a + \kappa\{f,g\}_b + \kappa^2\{f,g\}_c, \tag{3}$$

where κ is an arbitrary parameter,

$$\{f,g\}_a = -3\,\mathrm{tr}(X)\,\{f,g\}_1,$$

$$\{f,g\}_c = X_1(f)X_2(g) - X_1(g)X_2(f),$$

where the vector fields X_i are defined as follows:

$$X_1(f) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_{ii}}, \qquad X_2(f) = \{h, f\}_1,$$

and

$$\{f, g\}_b = X_3(\{f, g\}_1) - \{X_3(f), g\}_1 - \{f, X_3(g)\}_1,$$

where

$$X_3(f) = \sum_{i,j=1}^{3} G_{i,j} \frac{\partial f}{\partial x_{ij}}.$$

Here,

$$G_{1,1} = (-2x_{11}x_{23} + x_{22}x_{23} + 36x_{12}x_{32} + x_{23}x_{33}) + x_{31}(x_{11} - 2x_{22} + x_{33})\, g_2 + 9\, x_{21}x_{31}\, g_3,$$
$$G_{2,2} = -G_{1,1}, \qquad G_{3,3} = 0,$$
$$G_{1,2} = (x_{11}x_{13} + x_{13}x_{22} - 3\, x_{12}x_{23} - 2x_{13}x_{33}) + (3x_{12}x_{31} + 5x_{11}x_{32} - 4x_{22}x_{32} - x_{32}x_{33})\, g_2$$
$$-3(2x_{11}x_{31} - x_{22}x_{31} - 3x_{21}x_{32} - x_{31}x_{33})\, g_3,$$
$$G_{1,3} = 3x_{13}x_{23} - (x_{11} - x_{22})(x_{11} + x_{22} - 2x_{33})\, g_2 - 3x_{21}(x_{11} + x_{22} - 2x_{33})\, g_3,$$
$$G_{2,1} = -3(x_{21}x_{23} + 12x_{12}x_{31} + 4x_{11}x_{32} - 8x_{22}x_{32} + 4x_{32}x_{33}) - 6x_{21}x_{31}\, g_2,$$
$$G_{2,3} = 3(4x_{11}x_{12} + 4x_{12}x_{22} + x_{23}^2 - 8x_{12}x_{33}) + x_{21}(x_{11} + x_{22} - 2x_{33})\, g_2,$$
$$G_{3,1} = 2(x_{11}x_{21} + x_{21}x_{22} + 18x_{32}^2 - 2x_{21}x_{33}) - 3x_{31}^2\, g_2,$$
$$G_{3,2} = -(x_{11} - x_{22})(x_{11} + x_{22} - 2x_{33}) + 6x_{31}x_{32}\, g_2 - 9x_{31}^2\, g_3.$$

It is easy to verify that bracket (3) has three functionally independent cubic Casimir functions and has no Casimir functions of lower degree.

Two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are said to be compatible if

$$\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2$$

is a Poisson bracket for any λ. Let

$$C(\lambda) = C_0 + \lambda C_1 + \lambda^2 C_2 + \cdots$$

be a Casimir function for the bracket $\{\cdot,\cdot\}_\lambda$. Then the coefficients C_i commute with each other with respect to both brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$.

Let $\mathbf{a} = (a_1, \ldots, a_m)$ be an arbitrary constant vector. Then any linear Poisson bracket

$$\{y_i,\, y_j\} = C^p_{i,j}\, y_p, \qquad i, j = 1, \ldots, m \tag{4}$$

produces a constant bracket compatible with the initial linear one by the shift of arguments $y_i \to y_i + \lambda a_i$.

Consider now a quadratic Poisson bracket

$$\{y_i,\, y_j\} = r^{p,q}_{i,j}\, y_p y_q, \qquad i,j = 1,\dots,m. \tag{5}$$

The shift $y_i \mapsto y_i + \lambda a_i$ leads to a Poisson bracket of the form $\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\} + \lambda\{\cdot,\cdot\}_1 + \lambda^2\{\cdot,\cdot\}_2$. If the coefficient of λ^2 is equal to zero, then this formula defines a linear Poisson bracket $\{\cdot,\cdot\}_1$ compatible with (5). Thus, in the case of quadratic brackets, the vector $\mathbf{a}$ is not arbitrary but it should satisfy the overdetermined system of algebraic equations

$$r^{p,q}_{i,j}\, a_p a_q = 0, \qquad i,j = 1,\dots,m. \tag{6}$$

Such a vector and the corresponding shift of coordinates are called *admissible*. The admissible vectors are nothing but 0-dimensional symplectic leaves for the Poisson bracket (5). Any p-dimensional vector space of admissible vectors generates p pairwise compatible linear brackets. Each of them is compatible with quadratic bracket (5).

One can easily verify that the shift

$$x_{ij} \to x_{ij} + \delta_{i,j}\lambda \tag{7}$$

is admissible for bracket (3). The corresponding linear bracket is just (1). Apart from that, for generic κ, bracket (3) possesses 8 one-dimensional spaces of admissible vectors. The coordinates of these vectors are defined by an algebraic equation of degree eight. A.Odesskii communicates me that a quadratic Poisson bracket having such properties should be isomorphic to the elliptic bracket of the $q_{9,2}(\tau)$-type [2].

The Casimir functions of the pencil $\{f,g\} + \lambda\{f,g\}_1$, where $\{,\}$ is described by Proposition 1 and $\{,\}_1$ is defined by (1), can be obtained by (7) from the Casismir functions of the quadratic brackets. They have the following structure:

$$C_1(\lambda) = K_1, \qquad C_2(\lambda) = K_2 + \lambda Q_2, \qquad C_3(\lambda) = K_3 + \lambda h + Q_1\lambda^2.$$

Here K_i are Casimir functions for $\{f,g\}$, K_1, Q_2, Q_1 are Casimir functions for $\{f,g\}_1$ and the Hamiltonian h is given by (2).

In order to obtain the classical elliptic Calogero-Moser Hamiltonians, one should use the following classical limit of formulas (5):

$$\begin{array}{cc} x_{i+1,j+1} = q_i p_j, & x_{1,i+1} = p_i, \\ x_{1,1} = -\sum_{j=1}^{n-1} q_j p_j + \beta, & x_{i+1,1} = q_i\, x_{1,1}, \end{array} \tag{8}$$

where p_i nad q_i satisfy the standard constant Poisson bracket. One can verify that these are Darboux coordinates on the minimal symplectic leaf of the $\mathfrak{gl}_n$-Poisson bracket. This leaf is the orbit of the diagonal matrix $\mathrm{diag}(\beta, 0, 0, ..., 0)$, i.e. the set of all rank one matrices U with $\mathrm{trace}\,\mathrm{U} = \beta$.

After substitution (8) into (2), we get commuting polynomials in the canonical variables p_i, q_i. The element h becomes the Calogero-Moser Hamiltonian written in unusual coordinates, the element k vanishes, m converts to the integral of third degree in momenta that commutes with the Hamiltonian, and the Casimir functions c_i become constants.

To bring the Hamiltonian and the cubic integral to the standard Calogero-Moser form one has to apply a canonical transformation, where the transformation rule for the coordinates is defined by (3).

Remark. For the classical elliptic Calogero-Moser Hamiltonian with four particles ($N = 3$) a similar quadratic bracket of the $q_{16,3}(\tau)$-type exists. This bracket is compatible with the linear $\mathfrak{gl}_4$-Poisson bracket and generates the Calogero-Moser Hamiltonian exactly in the same way as for $N = 2$.

4 Conclusion

We have shown that the classical elliptic Calogero-Moser Hamiltonian with three particles is generated by the elliptic quadratic Poisson bracket of the $q_{9,2}(\tau)$-type in the frame of the bi-Hamiltonian approach. Namely, we obtain a pencil of compatible brackets by the shift of argument (7), find commuting polynomials as the coefficients of the Casimir functions for the pencil, restrict these polynomials to the minimal symplectic leaf of the linear bracket and perform an appropriate canonical transformation.

Possibly, the same procedure for the bracket of the $q_{n^2,n-1}(\tau)$-type gives rise to the Calogero-Moser Hamiltonian with n particles. Recently it was verified for $n = 4$.

A similar relation between an elliptic Poisson bracket and the elliptic Gaudin model was discovered in [6]. It would be interesting to understand which integrable Hamiltonian of the Calogero-Moser type corresponds to the elliptic bracket of $q_{9,5}(\tau)$-type.

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Bäcklund Transformations and New Integrable Systems on the Plane

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Abstract. The hyperelliptic curve cryptography is based on the arithmetic in the Jacobian of a curve. In classical mechanics well-known cryptographic algorithms and protocols can be very useful for construct auto-Bäcklund transformations, discretization of continuous flows and study of integrable systems with higher order integrals of motion. We consider application of a standard arithmetic of divisors on genus two hyperelliptic curve for the construction of new auto-Bäcklund transformations for the Hénon-Heiles system. Another type of auto-Bäcklund transformations associated with equivalence relations between unreduced divisors and the construction of the new integrable systems in the framework of the Jacobi method are also discussed.

Keywords: Bäcklund transformations · Integrable systems
Hyperelliptic curve cryptography

1 Introduction

The Jacobi method of separation of variables is a very important tool in analytical mechanics. According to [18], the stationary Hamilton-Jacobi equation

$$H = E$$

is said to be separable in a set of canonical coordinates $u = (u_1, \ldots, u_m)$ and $p_u = (p_{u_1}, \ldots, p_{u_m})$

$$\{u_i, p_{u_j}\} = \delta_{ij}\,, \qquad \{u_i, u_j\} = \{p_{u_i}, p_{u_j}\} = 0\,, \qquad i, j = 1, \ldots, m\,,$$

if there is an additively separated complete integral

$$W(u_1, \ldots, u_m; \alpha_1, \ldots, \alpha_m) = \sum_{i=1}^{m} W_i(u_i; \alpha_1, \ldots, \alpha_m)\,, \qquad \alpha_1, \ldots, \alpha_m \in \mathbb{R}\,,$$

depending non-trivially on a set of separation constants $\alpha_1, \ldots, \alpha_m$, and W_i are found in quadratures as solutions of ordinary differential equations.

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In the framework of the Jacobi method, after finding new variables of separation u, p_u on the phase space M, we have to look for new problems to which they can be successfully applied, see lecture 26 in [18]. Indeed, substituting the canonical coordinates into the separated relations

$$\Phi_i(u_i, p_{u_i}, H_1, \ldots, H_m) = 0\,, \qquad i = 1, \ldots, m, \qquad \text{with} \quad \det\left[\frac{\partial \Phi_i}{\partial H_j}\right] \neq 0 \quad (1)$$

and solving the resulting equations for $H_1, \ldots, H_m$, we obtain a new integrable system with independent integrals of motion $H_1, \ldots, H_m$ in involution.

In [26, 31–33] we proposed to add one more step to this well-known construction of integrable systems proposed by Jacobi. Namely, we can:

1. take the Hamilton-Jacobi equation $H = E$ separable in variables u, p_u;
2. make auto-Bäcklund transformation (BT) of the variables $(u, p_u) \to (\widetilde{u}, \widetilde{p}_u)$, which conserves not only the Hamiltonian character of the equations of motion, but also the form of Hamilton-Jacobi equation;
3. substitute the new canonical variables $\widetilde{u}, \widetilde{p}_u$ into suitable separated relations and obtain new integrable systems.

By definition auto-BT is a canonical transformation preserving the form of the Hamilton and the Hamilton-Jacobi equations, i.e. it is a symmetry of integrals of motion.

For many integrable by quadratures dynamical systems in holonomic and nonholonomic mechanics, the generic level set of integrals of motion can be related to the Jacobian of some hyperelliptic curve [6, 16, 35]. This allows us to study symmetries of integrals of motion using group operations in this Jacobian

$$D \approx D', \quad D + D' = D'' \qquad \text{and} \qquad [\ell]D = D''\,, \tag{2}$$

where $\approx$, $+$ and $[\ell]$ denote equivalence, addition and scalar multiplication by an integer of divisors D and D', respectively.

In [4], Cantor proposed an algorithm for performing computations in Jacobian groups of hyperelliptic curves which consists of two stages: the composition stage, which generally outputs an unreduced divisor, and the reduction stage which transforms the unreduced divisor into the unique reduced divisor. Now, we have plenty of algorithms and their professional computer implementations for the divisor arithmetics on low, mid and high-genus hyperelliptic curves, for generic divisor doubling, tripling, halving, for degenerate divisors operations and other special cases, in particular see [5, 8, 10, 12, 13, 23, 28].

In Hamiltonian mechanics, authors usually avoid to use the reduction stage of Cantor's algorithm and consider only full degree divisors D, D'' and weight one divisor D' in (2). This partial group operation was specially developed for the simplest one-parametric discretization of original continuous systems [7, 15, 21, 25].

Our main aim is to show explicitly an example of auto-BTs associated with the generic divisor doubling at $\ell = 2$ in (2) and discuss auto-BTs associated with equivalence relations. Such auto-BTs represent hidden symmetries of the level manifold which yields new canonical variables on the original phase space and, therefore, we can use these new variables to construct new integrable systems, i.e. systems of hetero-Bäcklund transformations.

2 Divisor Arithmetic on Hyperelliptic Curves

In this Section, we give some brief background on hyperelliptic curves and divisor arithmetics [12].

A hyperelliptic curve C of genus g is given by an equation

$$C: \qquad y^2 = f(x)\,, \qquad f(x) = a_{2g+1}x^{2g+1} + \cdots + a_1 x + a_0\,,$$

where $f(x)$ is a monic polynomial of degree $2g+1$ with distinct roots. Let $P(x,y)$ be a point on the curve C and P_∞ is the unique point at infinity on the projective closure of C. The point $-P = (x,-y)$ is called the opposite of P and P is a ramification point if $P = -P$ holds.

Divisor is a formal sum of points $P_i = (x_i, y_i)$

$$D = \sum_{P_i \in C} m_i P_i\,, \qquad m_i \in \mathbb{Z}\,,$$

where only a finite number of the m_i are non-zero. Support of D is the set $supp(D) = \{P_1|m_1, P_2|m_2 \ldots, P_k|m_k\}$ of all the points in a formal sum according to multiplicity, whereas the sum of multiplicities of points $deg(D) = \sum m_i$ is called the degree of a divisor.

The set of all divisors $\mathbb{D}$ forms an additive abelian group under the formal addition rule

$$\sum_{P_i \in C} m_i P_i + \sum_{P_i \in C} n_i P_i = \sum_{P_i \in C} (m_i + n_i) P_i\,.$$

Two divisors are equivalent if their difference is equal to the divisor of zeroes and poles of a rational function. i.e. a principal divisor:

$$\text{if} \quad D - D' \in \mathbb{P}\,, \quad \text{then} \quad D \approx D'\,.$$

Here $\mathbb{P}$ is the subgroup of $\mathbb{D}$ consisting of principal divisors. It is also a subgroup of $\mathbb{D}^0$, which is the subgroup of $\mathbb{D}$ consisting of the degree zero divisors only. The quotient group $Jac(C) = \mathbb{D}^0/\mathbb{P}$ is called the Jacobian of the curve C.

In order to uniquely represent divisor classes in the Jacobian we can use so-called reduced divisors. A semi-reduced divisor is a divisor of the form

$$D = \sum_{P_i \in C} m_i P_i - \left(\sum_{P_i \in C} m_i\right) P_\infty = E - mP_\infty\,, \qquad P_i \neq P_\infty\,,$$

where $m_i > 0$, $P_i \neq -P_j$ for $i \neq j$, no P_i satisfying $P_i = -P_i$ appears more than once and E is the so-called effective divisor. For each divisor $D \in \mathbb{D}_0$ there is a semi-reduced divisor $D_{sr} \in \mathbb{D}_0$ such that $D \approx D_{sr}$, but semi-reduced divisors are not unique in their equivalence class.

A semi-reduced divisor D is called reduced if $\sum m_i \leq g$ is no more than genus of curve C. This is a consequence of Riemann-Roch theorem for hyperelliptic curves that for each divisor $D \in \mathbb{D}_0$ there is a unique reduced divisor $\rho(D)$ such that $D \approx \rho(D)$. We drop the ρ from hereon and, unless stated otherwise, assume divisor Eq. (2) involve only reduced divisors.

So, every element of $Jac(C)$ can be represented by a unique reduced divisor as above, i.e. every semi-reduced divisor can be further reduced to a reduced divisor. This reduced representation of D is called the reduction of D and $m = \sum m_i$ called weight or reduced degree of D. We can employ the Mumford representation for the reduced and semi-reduced divisors $D = (U(x), V(x))$

$$U(x) = \prod_{P_i \in C} (x - x_i)^{m_i}\,, \qquad V(x_i) = y_i\,, \qquad deg(V) < deg(U) \leq g\,,$$

so that

$$V^2 - f \equiv 0 \mod U\,.$$

Here, the monic polynomial $U(x)$ may have multiple roots and $V(x)$ is tangent to the curve according to multiplicity roots.

Example 1. The unique divisor of zero weight $O = (1, 0)$ is a neutral element of the additional law. The divisor of weight one with a single point $P = (\lambda, \mu)$ on a curve is $D = (x-\lambda, \mu)$. The divisor of weight two is $D = (x^2+U_1x+U_0, V_1x+V_0)$, where

$$U(x) = x^2 + U_1x + U_0 = (x - x_1)(x - x_2)\,, \text{ if } \quad \mathrm{supp}(D) = \{P_1, P_2\}$$

$$U(x) = x^2 + U_1x + U_0 = (x - x_1)^2\,, \qquad \text{if} \quad \mathrm{supp}(D) = \{P_1|2\}$$

for the divisors with two and one points on a curve, respectively.

The Jacobian of a hyperelliptic curve is a group of degree zero divisors modulo principal divisors, and the group operation is formal addition modulo the equivalence relations. To compute the additive group law, Cantor gave a concrete algorithm which is applicable to a hyperelliptic curve of any genus [4]. The composition part of this algorithm computes the semi-reduced divisor $\widetilde{D} = D + D'$ that is equivalent to D''. The reduction part computes the reduced divisor D''.

Example 2. Let us consider an addition of two full degree divisors D and D' with Mumford coordinates

$$U(x) = \prod_{i=1}^{3} (x - x_i) \qquad \text{and} \qquad U'(x) = \prod_{i=1}^{3} (x - x'_i)$$

on the hyperelliptic genus three curve with $f(x) = a_7x^7 + \cdots + a_0$.

On the composition stage we compute the coefficients of the quintic polynomial $\mathscr{P}(x) = \sum_{k=0}^{5} b_kx^k$ that interpolates the six non-trivial points in the combined support of D and D'. Then, we compute abscissas $\widetilde{x}_k$ of the remaining four points of intersection using Abel polynomial

$$\psi(x) = f(x) - \mathscr{P}^2(x) = -b_5 \prod_{i=1}^{3}(x - x_i) \prod_{i=1}^{3}(x - x_i') \prod_{i=1}^{4}(x - \widetilde{x}_i)$$

and find the corresponding coordinates $\widetilde{y}_i = \mathscr{P}(\widetilde{x}_i)$ using $\mathscr{P}(x)$. As a result, we obtain an unreduced divisor $\widetilde{D} = \widetilde{P}_1 + \widetilde{P}_2 + \widetilde{P}_3 + \widetilde{P}_4 - 4P_\infty$.

On the reduction stage we are looking for the cubic function $U''(x)$ which interpolates the points in support of $\widetilde{D}$ and intersects C in three more places to form $D'' = P_1'' + P_2'' + P_3'' - 3P_\infty$. As a result, we obtain a reduced divisor D'' which is equivalent to $\widetilde{D}$ (Fig. 1).

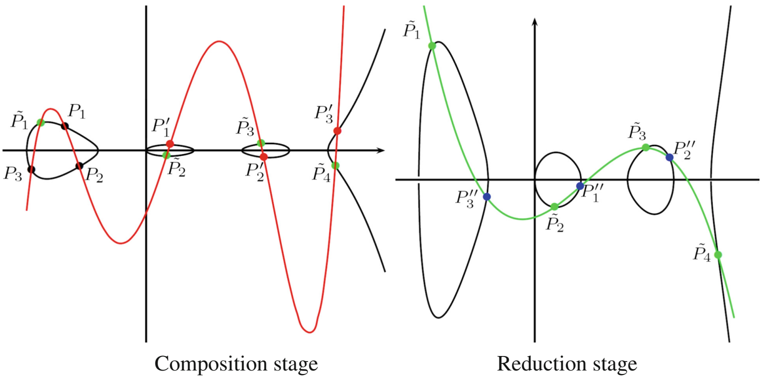


Fig. 1. Standard picture from the paper on hyperelliptic curve cryptography [5]

Example 3. In classical mechanics, we usually add divisors D and D' with Mumford coordinates

$$U(x) = \prod_{i=1}^{3}(x - x_i) \qquad \text{and} \qquad U'(x) = x - \lambda$$

so that

$$\psi(x) = f(x) - \mathscr{P}^2(x) = a_7 \prod_{i=1}^{3}(x - x_i)(x - \lambda) \prod_{i=1}^{3}(x - \widetilde{x}_i)\,,$$

where $\mathscr{P}(x)$ is the cubic polynomial that interpolates four points in the combined support of D and D'. On the composition stage, we form a divisor $\widetilde{D}$ with the order three from the remaining points of intersection, whereas the result of reduction is $-\widetilde{D}$.

Example 4. Let $D = (U(x), V(x)$ be a reduced divisor different from O. The operation

$$[2]D = D + D = D''$$

is a doubling of divisor D, i.e. scalar multiplication on integer $\ell = 2$. Its inverse is called halving of D'' and for a given D'' this equation has 2^{2g} solutions, any two of which differ by a 2-torsion divisor [10]. For an efficient implementation of generic divisor halving see [23] and references within. For instance, if $g = 2$ and $D = (x - \lambda, \mu)$ is a weight one divisor, then $[2]D = ((x - \lambda)^2, ax + b)$ where $ax + b$ is the tangent line at $P = (\lambda, \mu)$ with $a = f'(\lambda)2/\mu$ and $b = \mu - a\lambda$.

2.1 Some Explicit Formulae for Arithmetic on Genus 2 Hyperelliptic Curves

In cryptography, we have to distinguish between odd and even characteristic fields [12], however in classical mechanics we can always apply the substitution $y \to y - h(x)/2$ and study the genus two hyperelliptic curve which is defined by equation

$$C: \quad y^2 = f(x)\,, \qquad f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\,. \tag{3}$$

Let us consider the addition of a full degree reduced divisor

$$D: \quad \mathrm{supp}(D) = \{P_1, P_2\} \cup \{P_\infty\}\,, w(D) = 2$$

with another reduced divisor D' in the following cases

$$\begin{aligned}
&1. \quad D + D' = D''\,, w(D') = 2\,, \mathrm{supp}(D') = \{P_1', P_2'\} \cup \{P_\infty\}\,,\\
&2. \quad [2]D = D''\,, \qquad D' = D\,, \qquad \mathrm{supp}(D') = \{P_1, P_2\} \cup \{P_\infty\}\,,\\
&3. \quad D + D' = D''\,, w(D') = 1\,, \mathrm{supp}(D') = \{P_1'\} \cup \{P_\infty\}\,.
\end{aligned} \tag{4}$$

The result is full degree reduced divisor D'' with $\mathrm{supp}(D'') = \{P_1'', P_2''\} \cup \{P_\infty\}$ and $w(D'') = 2$. Below we do not consider halving, tripling and other operations because the resulting equations for the corresponding auto-Bäcklund transformations are quite bulky and unreadable.

In fact, there are two main methods for deriving group law in the Jacobian of hyperelliptic curve: the algebraic method based on Harley's implementation [10,13] of Cantor's algorithm [4], and the geometric method using interpolation of points [5], which is based on Clebsch's geometric formulation of the Abel theorem.

In the second case we add two divisors D and D' by finding a function which divisor contains the support of both D and D', and then the sum is equivalent to the negative of the compliment of that support. Following Abel's main idea from [1] such a function $\mathscr{P}(x)$ can be obtained by interpolating the points in the support of the two divisors. Compliment of the support D and D' in the support of $div(\mathscr{P})$ consists of other points of intersection of $y = \mathscr{P}(x)$ with the curve C.

Let us consider intersection of C with the second plane curve defined by equation

$$y - \mathscr{P}(x) = 0\,, \qquad \mathscr{P}(x) = b_3x^3 + b_2x^2 + b_1x + b_0\,. \tag{5}$$

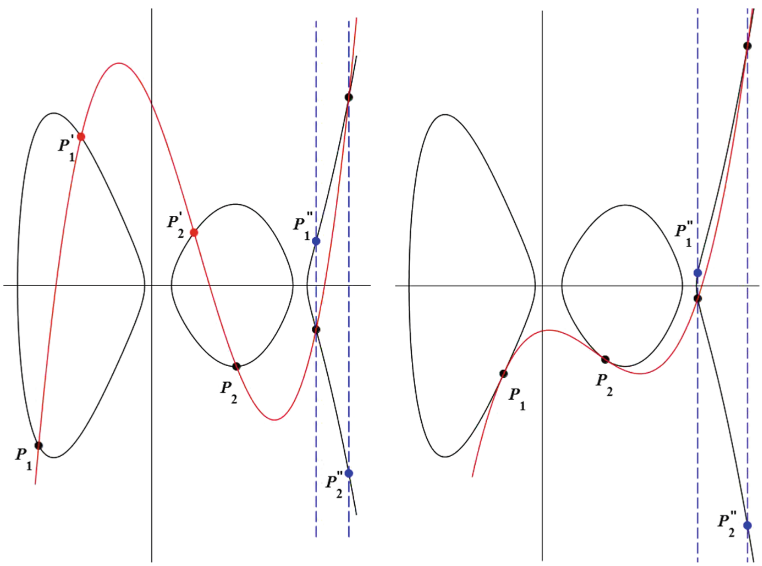
Substituting $y = \mathscr{P}(x)$ into the Eq. (3), we obtain the so-called Abel polynomial [1]

$$\psi(x) = \mathscr{P}(x)^2 - f(x)\,,$$

which has no multiple roots in the first and third cases and has double roots in the second case:

1. $\psi(x) = b_3^2(x-x_1)(x-x_2)(x-x_1')(x-x_2')(x-x_1'')(x-x_2'')\,,$
2. $\psi(x) = b_3^2(x-x_1)^2(x-x_1')^2(x-x_1'')(x-x_2'')\,,$
3. $\psi(x) = -a_5(x-x_1)(x-x_2)(x-x_1')(x-x_1'')(x-x_2'')\,, \quad b_3 = 0\,.$

The cases 1 and 2 are presented in Fig. 2.



Case 1: $(P_1+P_2)+(P_1'+P_2') = P_1''+P_2''$ Case 2: $[2](P_1+P_2) = P_1''+P_2''$

Fig. 2. The cases 1 and 2

Equating coefficients of ψ in the first case gives

$$x_1'' + x_2'' = -x_1 - x_2 - x_1' - x_2' + \frac{a_5 - 2b_2b_3}{b_3^2}\,, \tag{6}$$

$$x_1''x_2'' = \frac{2b_1b_3 + b_2^2 - a_4}{b_3^2} - (x_1 + x_2 + x_1' + x_2')(x_1'' + x_2'') - x_1(x_2 + x_1' + x_2') - x_2(x_1' + x_2') - x_1'x_2'\,.$$

In the second case we have to put $x'_{1,2}=x_{1,2}$ in these equations, whereas in the third case we have

$$
\begin{aligned}
x''_1+x''_2&=-x_1-x_2-x'_1+\tfrac{b_2^2-a_4}{a_5},\\
x''_1x''_2&=\tfrac{a_3-2b_1b_2}{a_5}-(x_1+x_2+x'_1)(x''_1+x''_2)-x_1x_2-x_1x'_1-x_2x'_1.
\end{aligned}
\tag{7}
$$

Four coefficients b_3, b_2, b_1 and b_0 of polynomial $\mathscr{P}(x)$ (5) are calculated by solving four algebraic equations:

$$
\begin{aligned}
&1.\ y_{1,2}=\mathscr{P}(x_{1,2}), \qquad y'_{1,2}=\mathscr{P}(x'_{1,2});\\
&2.\ y_{1,2}=\mathscr{P}(x_{1,2}), \qquad \frac{dP(x)}{dx}\Big|_{x=x_{1,2}}=\frac{d\sqrt{f(x)}}{dx}\Big|_{x=x_{1,2}}\equiv\frac{1}{2y_{1,2}}\,\partial f(x_{1,2}),\\
&3.\ y_{1,2}=\mathscr{P}(x_{1,2}), \qquad y'_1=\mathscr{P}(x'_1), \qquad b_3=0,
\end{aligned}
\tag{8}
$$

where $\partial f(x)$ is derivative of $f(x)$ (3) by x. Substituting coefficients b_k into (6)–(7) one gets abscissas $\widetilde{u}_{1,2}$, whereas the corresponding ordinates $y''_{1,2}$ are equal to

$$
y''_{3,4}=-\mathscr{P}(x''_{1,2}),
\tag{9}
$$

where polynomial $\mathscr{P}(x)$ is given by

$$
\begin{aligned}
1.\quad \mathscr{P}(x)&=\tfrac{(x-x'_2)(x-x'_1)(x-x_2)y_1}{(x_1-x'_1)(x_1-x'_2)(x_1-x_2)}+\tfrac{(x-x'_2)(x-x'_1)(x-x_1)y_2}{(x_2-x'_1)(x_2-x'_2)(x_1-x_2)}\\
&\quad+\tfrac{(x-x'_2)(x-x_2)(x-x_1)y'_1}{(x'_1-x_1)(x'_1-x_2)(x'_2-x'_1)}+\tfrac{(x-x'_1)(x-x_2)(x-x_1)y'_2}{(x'_2-x_1)(x'_2-x_2)(x'_2-x'_1)},\\
2.\quad \mathscr{P}(x)&=\tfrac{(x-x_2)^2(2x-3x_1+x_2)y_1}{(x_2-x_1)^3}+\tfrac{(x-x_1)^2(2x+x_1-3x_2)y_2}{(x_1-x_2)^3}\\
&\quad+\tfrac{(x-x_2)^2(x-x_1)\partial f(x_1)}{2(x_1-x_2)^2y_1}+\tfrac{(x-x_1)^2(x-x_2)\partial f(x_2)}{2(x_1-x_2)^2y_2},\\
3.\quad \mathscr{P}(x)&=\tfrac{y_1(x-x_2)(x-x'_1)}{(x_1-x_2)(x_1-x'_1)}+\tfrac{y_2(x-x_1)(x-x'_1)}{(x_2-x_1)(x_2-x'_1)}+\tfrac{y'_1(x-x_1)(x-x_2)}{(x'_1-x_1)(x'_1-x_2)}.
\end{aligned}
\tag{10}
$$

In (9) we also made a reduction, which coincides with inversion in our cases.

To perform such computations in generic cases we can take any professional implementation of Cantor's algorithm or implementations of more effective improvements and extensions to Cantor's algorithm for mid- or high-genus curves with general divisor doubling, tripling etc. [4,5,8,10,12,13,23,28].

3 Group Law and New Integrable Systems on the Plane

In this section we consider auto-BTs associated with addition and multiplication by an integer, which allows us to get new integrable deformations of two known integrable systems on the plane.

In Cantor's algorithm we use of the polynomial representation of group elements $D = (U(x), V(x)$ proposed by Mumford [24]. In fact, polynomials $U(x)$ and $V(x)$ were introduced by Jacobi in order to describe generating functions of algebraic integrals for Abel differential equations appearing in classical mechanics [17], see also comments in [24]. These polynomials define 2×2 Lax matrix for the corresponding integrable system

$$L(x) = \begin{pmatrix} V(x) & U(x) \\ \frac{V^2(x)-f(x)}{U(x)} & -V(x) \end{pmatrix}, \tag{11}$$

see [16,21,35] and references within. After a similarity transformation

$$L(x) \to \widetilde{L}(x) = ML(x)M^{-1}, \qquad M = \begin{pmatrix} U(x) & 0 \\ V(x) - \mathscr{P}(x) & U(x) \end{pmatrix}$$

one gets another Lax matrix

$$\widetilde{L}(x) = \begin{pmatrix} \mathscr{P}(x) & U(x) \\ -\psi(x)U^{-1} & -\mathscr{P}(x) \end{pmatrix}. \tag{12}$$

Zeroes of two off-diagonal elements of this matrix are variables of separation related by the desired auto-BT.

The philosophy advocated in [15,21,25,27] requires that:

- Lax matrix $\widetilde{L}(x)$ has the same form as the original Lax matrix $L(x)$;
- gauge matrix M has to satisfy the r-matrix Poisson bracket with the same r-matrix as the original Lax matrix $L(x)$.

In the framework of the Abel theory, we prefer to abandon this philosophy, because form of $\widetilde{L}(x)$ (12) is different from the form of the original Lax matrix $L(x)$ (11).

In hyperelliptic cryptography for affine coordinates associated with Mumford representation each group operation needs one reduction. Using various projective coordinates (weighted, Jacobian, modified Jacobian, mixed, etc.) we can avoid reductions on the cost of more multiplications. Moreover, for fixed genus, one can make the steps of the algorithm explicit and a more clever ordering results in faster formulae for addition and doubling. Summing up, we have to use different coordinates for different purpose.

In the similar manner we can use similarity transformation of Lax matrices (11) for discretization of continuous Hamiltonian flows in terms of original coordinates, but Jacobi construction of new integrable systems needs variables of separation, which are abscissas and ordinates of points P_i on curve C. In this approach auto-BTs on genus two hyperelliptic curve are completely defined by explicit formulae (6), (7) and (9).

3.1 Hénon-Heiles System

Let us take Hénon-Heiles system with Hamiltonians

$$H_1 = \frac{p_1^2 + p_2^2}{4} - 4aq_2(q_1^2 + 2q_2^2), \qquad H_2 = \frac{p_1(q_1p_2 - q_2p_1)}{2} - aq_1^2(q_1^2 + 4q_2^2) \tag{13}$$

separable in parabolic coordinates on the plane

$$u_1 = q_2 - \sqrt{q_1^2 + q_2^2}, \qquad u_2 = q_2 + \sqrt{q_1^2 + q_2^2}\,. \tag{14}$$

Standard momenta associated with parabolic coordinates $u_{1,2}$ are equal to

$$p_{u_1} = \frac{p_2}{2} - \frac{p_1(q_2 + \sqrt{q_1^2 + q_2^2})}{2q_1}\,, \qquad p_{u_2} = \frac{p_2}{2} - \frac{p_1(q_2 - \sqrt{q_1^2 + q_2^2})}{2q_1}\,.$$

To describe evolution of $u_{1,2}$ with respect to $H_{1,2}$ we use the canonical Poisson bracket

$$\begin{gathered}\{q_i, p_j\} = \delta_{ij}\,, \quad \{q_1, q_2\} = \{p_1, p_2\} = 0\,, \\ \{u_i, p_{u_j}\} = \delta_{ij}\,, \quad \{u_1, u_2\} = \{p_{u_1}, p_{u_2}\} = 0\,.\end{gathered} \tag{15}$$

and expressions for $H_{1,2}$

$$\begin{aligned} H_1 &= \tfrac{p_{u_1}^2 u_1 - p_{u_2}^2 u_2}{u_1 - u_2} - a(u_1 + u_2)(u_1^2 + u_2^2)\,, \\ H_2 &= \tfrac{u_1 u_2 (p_{u_1}^2 - p_{u_2}^2)}{u_2 - u_1} + a u_1 u_2 (u_1^2 + u_1 u_2 + u_2^2) \end{aligned} \tag{16}$$

to obtain

$$\frac{du_1}{dt_1} = \{u_1, H_1\} = \frac{2p_{u_1} u_1}{u_1 - u_2}\,, \qquad \frac{du_2}{dt_1} = \{u_2, H_1\} = \frac{2p_{u_2} u_2}{u_2 - u_1} \tag{17}$$

and

$$\frac{du_1}{dt_2} = \{u_1, H_2\} = \frac{2u_1 u_2 p_{u_1}}{u_2 - u_1}\,, \qquad \frac{du_2}{dt_2} = \{u_2, H_2\} = \frac{2u_1 u_2 p_{u_2}}{u_1 - u_2}\,. \tag{18}$$

Using Hamilton-Jacobi equations $H_{1,2} = \alpha_{1,2}$ we can prove that these variables satisfy to the following separated relations

$$\big(u_i p_{u_i}\big)^2 = u_i(a u_i^4 + \alpha_1 u_i + \alpha_2)\,, \qquad i = 1, 2. \tag{19}$$

Expressions (17), (18) and (19) yield standard Abel quadratures

$$\frac{du_1}{\sqrt{f(u_1)}} + \frac{du_2}{\sqrt{f(u_2)}} = 2dt_2\,, \qquad \frac{u_1 du_1}{\sqrt{f(u_1)}} + \frac{u_2 du_2}{\sqrt{f(u_2)}} = 2dt_1, \tag{20}$$

on hyperelliptic curve C of genus two defined by equation

$$C: \quad y^2 = f(x)\,, \qquad f(x) = x(ax^4 + \alpha_1 x + \alpha_2)\,. \tag{21}$$

Suppose that transformation of variables

$$(q_1, q_2, p_1, p_2) \to (\widetilde{q}_1, \widetilde{q}_2, \widetilde{p}_1, \widetilde{p}_2) \tag{22}$$

preserves Hamilton equations (17) and (18) and the form of Hamiltonians (16). It means that new parabolic coordinates $\widetilde{u}_{1,2} = \widetilde{q}_2 \pm \sqrt{\widetilde{q}_1^2 + \widetilde{q}_2^2}$ satisfy to the same equations

$$\frac{d\widetilde{u}_1}{\sqrt{f(\widetilde{u}_1)}} + \frac{d\widetilde{u}_2}{\sqrt{f(\widetilde{u}_2)}} = 2dt_2\,, \qquad \frac{\widetilde{u}_1 d\widetilde{u}_1}{\sqrt{f(\widetilde{u}_1)}} + \frac{\widetilde{u}_2 d\widetilde{u}_2}{\sqrt{f(\widetilde{u}_2)}} = 2dt_1\,. \tag{23}$$

Subtracting (23) from (20) one gets Abel differential equations

$$\begin{aligned} &\omega_1(x_1,y_1) + \omega_1(x_2,y_2) + \omega_1(x_1'',y_1'') + \omega_1(x_2'',y_2'') = 0\,, \\ &\omega_2(x_1,y_1) + \omega_2(x_2,y_2) + \omega_2(x_2'',y_1'') + \omega_2(x_2'',y_2'') = 0\,, \end{aligned} \tag{24}$$

where

$$x_{1,2} = u_{1,2}, \quad y_{1,2} = u_{1,2}p_{u_{1,2}}\,, \qquad x''_{1,2} = \widetilde{u}_{1,2}, \quad y''_{1,2} = -\widetilde{u}_{1,2}\widetilde{p}_{u_{1,2}}$$

and $\omega_{1,2}$ form a base of holomorphic differentials on hyperelliptic curve C of genus $g = 2$

$$\omega_1(x,y) = \frac{dx}{y}\,, \qquad \omega_2(x,y) = \frac{xdx}{y}\,.$$

These Abel equations are related to the Picard group which is isomorphic to Jacobian of C [20] and we can consider canonical transformation (22) of variables in the phase space as some group operation in the Jacobian.

Let us suppose that generic points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ form divisor D in (4), whereas points $P''_{1,2} = (x''_{1,2}, y''_{1,2})$ belong to support of resulting divisor D''. Canonical transformations (22) associated with arithmetic operations (4) are completely defined by coefficients b_k of polynomial $\mathscr{P}$ (10). In the first case, these coefficients b_k are defined by

$$\begin{aligned} Ab_0 &= q_1x_1'x_2'(x_1'-x_2')\Bigl(p_1\bigl(q_1^2+4q_2^2-2q_2(x_1'+x_2')+x_1'x_2'\bigr)+q_1p_2(x_1'+x_2'+2q_2)\Bigr) \\ &- 2q_1^2x_2'(q_1^2+2q_2x_2'-{x_2'}^2)y_1'+2q_1^2x_1'(q_1^2+2q_2x_1'-{x_2'}^2)y_2'; \\ Ab_1 &= (x_2'-x_1')\Bigl(q_1p_1\bigl((q_1^2+4q_2^2)(x_1'+x_2')-2q_2({x_1'}^2+x_1'x_2'+{x_2'}^2)\bigr) \\ &\qquad - p_2\bigl(2q_1^2q_2(x_1'+x_2')-q_1^2({x_1'}^2+x_1'x_2'+{x_2'}^2)+{x_1'}^2{x_2'}^2\bigr)\Bigr) \\ &+ 2(q_1^2-2q_2x_2')(q_1^2+2q_2x_2'-{x_2'}^2)y_1'-2(q_1^2-2q_2x_1')(q_1^2+2q_2x_1'-{x_1'}^2)y_2'\,, \\ Ab_2 &= (x_1'-x_2')\Bigl(q_1p_1(q_1^2+4q_2^2-{x_1'}^2-x_1'x_2'-{x_2'}^2)-\bigl(2q_1^2q_2+(x_1'+x_2')x_1'x_2'\bigr)p_2\Bigr) \\ &+ 2(2q_2+x_2')(q_1^2+2q_2x_2'-{x_2'}^2)y_1'-2(2q_2+x_1')(q_1^2+2q_2x_1'-{x_1'}^2)y_2'\,, \end{aligned}$$

$$Ab_3 = (x_2'-x_1')\bigl(q_1p_1(2q_2-x_1'-x_2')-(q_1^2+x_1'x_2')p_2\bigr)-2(q_1^2+2q_2x_2'-{x_2'}^2)y_1'-2(q_1^2+2q_2x_1'-{x_1'}^2)y_2'\,,$$

where

$$A = 2(x_1'-x_2')(q_1^2+2q_2x_1'-{x_2'}^2)(q_1^2+2q_2x_2'-{x_2'}^2).$$

Substituting these coefficients into the following expressions one gets explicit formulae for the new variables

$$
\begin{aligned}
&\widetilde{q}_2 = -q_2 - \tfrac{x_1'+x_2'}{2} - \tfrac{b_2}{b_3} + \tfrac{a}{2b_3^2}\,,\\
&\widetilde{q}_1^2 = -q_1^2 + 2\widetilde{q}_2(2q_2 + x_1' + x_2') + 2q_2(x_1' + x_2') + x_1'x_2' - \tfrac{2b_1}{b_3} - \tfrac{b_2^2}{b_3^2}\,,\\
&\widetilde{p}_1 = -\tfrac{2b_0}{\widetilde{q}_1} - 4\widetilde{q}_1\widetilde{q}_2 b_3 - 2\widetilde{q}_1 b_2\,, \quad \widetilde{p}_2 = -2(\widetilde{q}_1^2 + 4\widetilde{q}_2^2)b_3 - 4\widetilde{q}_2 b_2 - 2b_1\,.
\end{aligned}
\tag{25}
$$

In Case 2 we have

$$
\begin{aligned}
Bb_0 &= -8aq_1^4(2p_1q_1q_2 - p2q_1^2 - 2p_2q_2^2) + q_1^2p_1(p_1^2q_1 + 2p_1p_2q_2 - 2p2^2q_1)\,,\\
Bb_1 &= -8aq_1^2(p_1q_1^3 + 10p_1q_1q_2^2 - 4p_2q_1^2q_2 - 4p_2q_2^3) - 2p_1^3q_1q_2 + 3p_1^2q_1^2p_2\\
&\quad - 4p_1^2p_2q_2^2 + 4p_1p_2^2q_1q_2 - 2p_2^3q_1^2\,,\\
Bb_2 &= -8aq_1(12p_1q_2^3 + p_2q_1^3 - 2p_2q_1q_2^2) + p_1^2(p_1q_1 + 4p_2q_2)\,,\\
Bb_3 &= 8aq_1(p_1q_1^2 + 6p_1q_2^2 - 2p_2q_1q_2) - p_2p_1^2\,,\\
B &= 4q_1(p_1^2q_1 + 2p_1p_2q_2 - p_2^2q_1)
\end{aligned}
$$

so that

$$
\begin{aligned}
&\widetilde{q}_2 = -2q_2 + \tfrac{1}{\left(8q_1(p_1q_1^2+6p_1q_2^2-2p_2q_1q_2)a-p_2p_1^2\right)^2}\\
&\times\Big(64q_1^2(12p_1q_2^3+p_2q_1^3-2p_2q_1q_2^2)(p_1q_1^2+6p_1q_2^2-2p_2q_1q_2)a^2+p_1^4p_2(p_1q_1+4p_2q_2)\\
&-8aq_1(6p_1^4q_1q_2^2-2p_1^3p_2q_1^2q_2+36p_1^3p_2q_2^3+3p_1^2p_2^2q_1^3-14p_1^2p_2^2q_1q_2^2+4p_1p_2^3q_1^2q_2-p_2^4q_1^3)\Big)\,,\\
&\widetilde{q}_1^2 =-\left(\tfrac{8aq_1^2(2p_1q_1q_2-p_2q_1^2-2p_2q_2^2)-p_1(p_1^2q_1+2p_1p_2q_2-2p_2^2q_1}{8aq_1(p_1q_1^2+6p_1q_2^2-2p_2q_1q_2)-p_2p_1^2}\right)^2,\\
&\widetilde{q}_1\widetilde{p}_1 = -2b_0 - 2\widetilde{q}_1^2(b_2 + 2\widetilde{q}_2 b_3)\,, \quad \widetilde{p}_2 = -2(\widetilde{q}_1^2 + 4\widetilde{q}_2^2)b_3 - 4\widetilde{q}_2 b_2 - 2b_1\,.
\end{aligned}
\tag{26}
$$

In Case 3 coefficients are equal to

$$
\begin{aligned}
b_0 &= \tfrac{(x_1'(2p_1q_2x_1'-p_2q_1)p_1{x_1'}^2+2q_1y_1')q_1}{2(q_1^2+2q_2x_1'-{x_1'}^2)}\\
b_1 &= -\tfrac{2p_1q_1q_2-p_2(q_1^2-{x_1'}^2)-4q_2y_1'}{2(q_1^2+2q_2x_1'-{x_1'}^2)}\,, \qquad b_2 = \tfrac{p_1q_1+p_2x_1'-2y_1'}{2(q_1^2+2q_2x_1'-{x_1'}^2)}\,,
\end{aligned}
\tag{27}
$$

so that

$$
\begin{aligned}
\widetilde{q}_2 &= -q_2 - \frac{x_1'}{2} + \frac{b_2^2}{2a} = -q_2 - \frac{x_1'}{2} + \frac{(p_1q_1 + p_2x_1' - 2y_1')^2}{8a(q_1^2 + 2q_2x_1' - {x_1'}^2)^2}\,,\\
\widetilde{q}_1^2 &= -q_1^2+2\widetilde{q}_2(2q_2+x_1')+2q_2x_1'+\tfrac{2b_1b_2}{a}\\
&= -q_1^2+2\widetilde{q}_2(2q_2+x_1')+2q_2x_1'-\tfrac{(p_1q_1+p_2x_1'-2y_1')(2p_1q_1q_2-p_2q_1^2+p_2{x_1'}^2-4q_2y_1')}{2a(q_1^2+2q_2x_1'-{x_1'}^2)^2}\,,\\
\widetilde{p}_2 &= -4\widetilde{q}_2b_2 - 2b_1 = -p_2 - \frac{2(q_2 - \widetilde{q}_2)(2y_1' - p_1q_1 - p_2x_1')}{q_1^2 + 2q_2x_1' - {x_1'}^2}\,,\\
\widetilde{q}_1\widetilde{p}_1 &= -2b_0 - 2q_1^2b_2 = -q_2p_2 - \frac{(q_1^2 - \widetilde{q}_2^2)(2y_1' - p_1q_1 - p_2x_1')}{q_1^2 + 2q_2x_1' - {x_1'}^2}\,.
\end{aligned}
\tag{28}
$$

These explicit formulae for $\widetilde{q}_{1,2}$ and $\widetilde{p}_{1,2}$ can be easily obtained using any modern computer algebra system.

We present these bulky expressions here only so that any reader can verify that these transformations $(q,p) \to (\widetilde{q},\widetilde{p})$ are different, i.e. cannot be obtained from each other for special values of parameters, and that these transformations have the following properties.

Theorem 1. *Equations (25), (26) and (28) determine canonical transformations (22) on $T^*\mathbb{R}^2$ of valencies one and two for which original Poisson bracket (15) has the following form in new variables*

$$1,3.\ \{\widetilde{q}_i,\widetilde{p}_j\} = \ \delta_{i,j}\,, \qquad \{\widetilde{q}_1,\widetilde{q}_2\} = \{\widetilde{p}_1,\widetilde{p}_2\} = 0\,,$$

$$2.\ \ \{\widetilde{q}_i,\widetilde{p}_j\} = 2\delta_{i,j}\,, \qquad \{\widetilde{q}_1,\widetilde{q}_2\} = \{\widetilde{p}_1,\widetilde{p}_2\} = 0\,,$$

respectively. These canonical transformations preserve the form of integrals of motion (13), i.e. they are auto-Bäcklund transformations of the Hénon-Heiles system.

The proof is a straightforward calculation. In the same manner, i.e. using a computer implementation of Cantor's algorithm, we can prove that auto-BT associated with tripling $D'' = [3]D$ of divisor has valence three.

For canonical transformation $(q,p) \to (\tilde{q},\tilde{p})$ of valence c the Jacobi matrix of transformation

$$V = \begin{pmatrix} \frac{\partial\tilde{q}}{\partial q} & \frac{\partial\tilde{q}}{\partial p} \\ \\ \frac{\partial\tilde{p}}{\partial q} & \frac{\partial\tilde{p}}{\partial p} \end{pmatrix}$$

is a generalized symplectic matrix of valence c

$$V^\top \varOmega V = c\varOmega\,, \qquad \varOmega = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix},$$

see details in [9].

Summing up, using Jacobian arithmetic for hyperelliptic curves, we can identify various cryptographic algorithms and protocols [12] with various schemes of the discretization of continuous Hamiltonian flows in classical mechanics. The fact most interesting to us is that the corresponding auto-BTs also yield new canonical variables on the original phase space, which can be used for construction of new integrable systems in the framework of the Jacobi method.

For instance, let us suppose that divisor D' in Case 3 consists of ramification point $P_0 = (0,0)$ on C. In this case variables $\widetilde{u}_{1,2}, \widetilde{p}_{u_{1,2}}$ are defined by Eqs. (7) and (9) for $x''_k = \widetilde{u}_k$ and $y''_k = \widetilde{u}_k\widetilde{p}_{u_k}$:

$$\widetilde{u}_1 + \widetilde{u_2} = -u_1 - u_2 + \tfrac{b_2^2}{a}, \quad \widetilde{u}_1\widetilde{u}_2 = -\tfrac{2b_1b_2}{a} - (u_1+u_2)(\widetilde{u}_1+\widetilde{u}_2) - u_1u_2,$$

$$\widetilde{p}_{u_1} = -\tfrac{b_2\widetilde{u}_1^2 + b_1\widetilde{u}_1 + b_0}{\widetilde{u}_1}\,, \qquad \widetilde{p}_{u_2} = -\tfrac{b_2\widetilde{u}_2^2 + b_1\widetilde{u}_2 + b_0}{\widetilde{u}_2}\,,$$

where coefficients b_k are given by (27). Substituting $y=\widetilde{p}_{1,2}$ and $x=\widetilde{u}_{1,2}$ into the separated relation

$$\widetilde{C}: \quad (y^2-ax^3-\widetilde{H}_1-\widetilde{H}_2)(y^2-ax^3-\widetilde{H}_1+\widetilde{H}_2)+abx+acy=0\,,$$

which defines genus three hyperelliptic curve $\widetilde{C}$, and solving the resulting equations with respect to $\widetilde{H}_{1,2}$, one gets Hamiltonian

$$\widetilde{H}_1=\frac{p_1^2}{8}+\frac{p_2^2}{4}-aq_2(3q_1^2+8q_2^2)+\frac{b}{2q_1^2}-\frac{cp_1}{q_1^3}\,. \tag{29}$$

Theorem 2. *After additional canonical transformation*

$$q_1\to q_1/\sqrt{2}\,,\quad p_1\to\sqrt{2}(p_1+2cq_1^{-3})$$

we identify (29) with Hamiltonian for second integrable Hénon-Heiles system with quartic additional integral H_2 [14].

Other integrable deformations of the original Hamiltonians (13) may be found in [31,32].

3.2 System with Quartic Potential

Let $q_{1,2}$ are Cartesian coordinates on the plane, then elliptic coordinates $u_{1,2}$ with parameters $\xi_{1,2}$ are defined through the equation

$$1+\frac{q_1^2}{u-\xi_1}+\frac{q_2^2}{u-\xi_2}=\frac{(u-u_1)(u-u_2)}{(u-\xi_1)(u-\xi_2)}\,,\qquad \xi_i\in\mathbb{R}\,.$$

The elliptic coordinate system is orthogonal, and the coordinates take values C only in the intervals

$$u_1<\xi_1<u_2<\xi_2\,.$$

By a simultaneous shifting of the coordinates and the parameters it is always possible to take $\xi_1=0$ and $\xi_2=\kappa^2>0$.

Substituting

$$q_1=\sqrt{\frac{(u_2-\xi_1)(\xi_1-u_1)}{\xi_2-\xi_1}},\qquad q_2=\sqrt{\frac{(\xi_2-u_1)(\xi_2-u_2)}{\xi_2-\xi_1}} \tag{30}$$

and momenta

$$p_1=\frac{2\Big(p_{u_1}(\xi_2-u_1)-p_{u_2}(\xi_2-u_2)\Big)\sqrt{u_2-\xi_1}\sqrt{\xi_1-u_1}}{(u_1-u_2)\sqrt{\xi_2-\xi_1}}$$

$$p_2=\frac{2\Big(p_{u_1}(\xi_1-u_1)-p_{u_2}(\xi_1-u_2)\Big)\sqrt{\xi_2-u_2}\sqrt{\xi_2-u_1}}{(u_1-u_2)\sqrt{\xi_2-\xi_1}}$$

in the Hamiltonian

$$H_1=\frac{p_1^2+p_2^2}{2}+V(q_1,q_2)$$

with potential

$$V = 2a(q_1^2+q_2^2-\xi_1-\xi_2)\Big((q_1^2+q_2^2)^2-2\xi_1q_1^2-2\xi_2q_2^2+\xi_1^2+\xi_2^2\Big)$$
$$-\ 2b\Big((q_1^2+q_2^2)^2-(2\xi_1+\xi_2)q_1^2-(\xi_1+2\xi_2)q_2^2+\xi_1^2+\xi_1\xi_2+\xi_2^2\Big)+2c(q_1^2+q_2^2-\xi_1-\xi_2),$$

one gets

$$H_1 = \frac{2(\xi_2-u_1)(\xi_1-u_1)p_{u_1}^2+2(\xi_2-u_2)(u_2-\xi_1)p_{u_2}^2}{u_2-u_1}+\frac{U(u_1)-U(u_2)}{u_1-u_2},$$

where

$$U = 2u^2(au^2+bu+c)\,.$$

According to Bertrand-Darboux theorem second integral of motion is equal to

$$H_2 = \frac{2u_2(\xi_2-u_1)(\xi_1-u_1)p_{u_1}^2+2u_1(\xi_2-u_2)(u_2-\xi_1)p_{u_2}^2}{u_2-u_1}+\frac{u_2U(u_1)-u_1U(u_2)}{u_2-u_1}\,,$$

so that separated relations have the form

$$(\xi_2-u_i)(\xi_1-u_i)p_{u_i}^2 = au_i^4+bu_i^3+cu_i^2+\alpha_1u_i+\alpha_2\,,\qquad i=1,2, \tag{31}$$

where $H_{1,2}=-2\alpha_{1,2}$. The corresponding Abel quadratures on the genus 2 hyperelliptic curve C reads as

$$\frac{du_1}{\sqrt{f(u_1)}}+\frac{du_2}{\sqrt{f(u_2)}}=4dt_1\,,\qquad \frac{u_1du_1}{\sqrt{f(u_1)}}+\frac{u_2du_2}{\sqrt{f(u_2)}}=4dt_2,$$

where $f(x)$ is the following polynomial of six order

$$f(u) = (u-\xi_1)(u-\xi_2)(au^4+bu^3+cu^2+\alpha_1u+\alpha_2)\,. \tag{32}$$

In order to get auto-Bäcklund transformation

$$B:\qquad (u_1,p_{u_1},u_2,p_{u_2})\to(\widetilde{u}_1,\widetilde{p}_{u_1},\widetilde{u}_2,\widetilde{p}_{u_2})$$

we can use Abel differential Eq. (24), where

$$x_{1,2}=u_{1,2}\,,\quad y_{1,2}=\ \ (u_{1,2}-\xi_1)(u_{1,2}-\xi_2)p_{u_{1,2}}$$
$$x_{3,4}=\widetilde{u}_{1,2}\,,\quad y_{3,4}=-(\widetilde{u}_{1,2}-\xi_1)(\widetilde{u}_{1,2}-\xi_2)\widetilde{p}_{u_{1,2}}\,.$$

For the brevity we present expressions for new coordinates only at $\lambda=0$ and $\xi_1=0$, $\xi_2=\kappa^2$:

$$\widetilde{u}_1+\widetilde{u}_2 = \frac{(u_1-u_2)^2\Big(a(u_1^2+u_2^2-\kappa^2u_1-\kappa^2u_2)+b(u_1+u_2-\kappa^2)+c\Big)+A}{(u_1-u_2)B}\,,$$

$$\widetilde{u}_1\widetilde{u}_2 = \frac{\kappa^2\Big((u_1-u_2)\Big(a(u_1^2+u_1u_2+u_2^2)+b(u_1+u_2)+c\Big)+(\kappa^2-u_1)p_{u_1}^2-(\kappa^2-u_2)p_{u_2}^2\Big)}{B},$$

where

$$A = 2\sqrt{a}(u_1-u_2)\Big(u_2(\kappa^2-u_1)p_{u_1}-u_1(\kappa^2-u_2)p_{u_2}\Big)-\Big((\kappa^2-u_1)p_{u_1}-(\kappa^2-u_2)p_{u_2}\Big)^2$$

and

$$B = a(u_1-u_2)(\kappa^2-2u_1-2u_2)+2\sqrt{a}\Big((\kappa^2-u_1)p_{u_1}-(\kappa^2-u_2)p_{u_2}\Big)-b(u_1-u_2).$$

The corresponding momenta are defined by polynomial

$$\mathscr{P}(x) = \sqrt{a}(x-u_1)(x-u_2)x+\frac{x(x-u_2)(\kappa^2-u_1)p_{u_1}}{u_1-u_2}+\frac{x(x-u_1)(\kappa^2-u_2)p_{u_2}}{u_2-u_1}\,.$$

Substituting these variables into the new separated equations

$$\Phi_\pm = (\xi-\widetilde{u})\widetilde{p}_u^2 - a\widetilde{u}^3 - b\widetilde{u}^2 - c\widetilde{u} + \frac{\widetilde{H}_1}{2} \pm \frac{\sqrt{\widetilde{H}_2}}{2} = 0\,, \qquad i=1,2\,. \tag{33}$$

one gets a pair of new Hamiltonians

$$\widetilde{H}_1 = H_1 + \frac{H_2}{2}\,\frac{\widetilde{u}_1+\widetilde{u}_2}{\widetilde{u}_1\widetilde{u}_2}\,, \qquad \sqrt{\widetilde{H}_2} = H_2\,\frac{\widetilde{u}_2-\widetilde{u}_1}{\widetilde{u}_1\widetilde{u}_2}\,.$$

which define new system of Hamilton-Jacobi equations. In original coordinates after scaling transformation

$$p_1 \to \sqrt{2}p_1\,, \qquad q_1 \to \frac{q_1}{\sqrt{2}},$$

first Hamiltonian is equal to

$$\begin{aligned}\widetilde{H}_1 =& \tfrac{p_1^2+p_2^2}{2}+a\left(2\kappa^6-(q_1^2+6q_2^2)\kappa^4+\left(\tfrac{3q_1^4}{4}+\tfrac{7q_1^2q_2^2}{2}+6q_2^4\right)\kappa^2-\tfrac{(q_1^2+4q_2^2)(q_1^2+2q_2^2)^2}{8}\right)\\ &+\sqrt{a}q_1\Big((\kappa^2-q_2^2)p_1+\tfrac{p_2q_1q_2}{2}\Big)+b\left(\tfrac{q_1^4}{4}+\tfrac{3q_1^2q_2^2}{2}+2q_2^4-\kappa^2(q_1^2+4q_2^2)+2\kappa^4\right)\\ &-c\left(\tfrac{q_1^2}{2}+2q_2^2-2\kappa^2\right).\end{aligned} \tag{34}$$

Theorem 3. *For $a \neq 0$ integrable Hamiltonian (34) with velocity dependent potential determines new integrable deformation of the known integrable Hamiltonian with quartic potential [14].*

Second Hamiltonian $\widetilde{H}_2$ is the polynomial of fourth order in the momenta. For brevity, we present this polynomial only for $b=c=0$

$$\begin{aligned}\widetilde{H}_2 =& \tfrac{p_1^4}{4} - \tfrac{\sqrt{a}p_1q_1\left(2(q_2^2-\kappa^2)p_1^2-3p_1p_2q_1q_2+p_2^2q_1^2\right)}{2} - \tfrac{aq_1^2}{8}\Big(\Big(q_1^4+12q_1^2q_2^2+12q_2^4-6\kappa^2(q_1^2-2q_2^2)\Big)p_1^2\\ &-8q_1q_2(q_1^2+q_2^2)p_1p_2+2q_1^2(q_1^2+q_2^2-\kappa^2)p_2^2\Big)-\tfrac{a^{3/2}q_1^3}{8}\Big(\Big(16\kappa^6+2(q_1^2+2q_2^2)(q_1^2q_2^2+2q_2^4-6\kappa^4)\\ &+2q_1^2(q_1^2+6q_2^2)\kappa^2\Big)p_1-q_1q_2\Big(q_1^4+4q_1^2q_2^2+4q_2^4+2\kappa^2(q_1^2-2q_2^2)\Big)p_2\Big)+\tfrac{a^2q_1^4}{64}\Big(64\kappa^8-32(3q_1^2+8q_2^2)\kappa^6\\ &+4(13q_1^4+60q_1^2q_2^2+84q_2^4)\kappa^4-4(3q_1^2+10q_2^2)(q_1^2+2q_2^2)^2\kappa^2+(q_1^2+2q_2^2)^4\Big).\end{aligned}$$

4 Equivalence Relations and New Integrable Systems on the Plane

Let us take an integrable system with Hamiltonians

$$
\begin{aligned}
H_1 &= \tfrac{p_1^2+p_2^2}{4} + \tfrac{a}{q_1^2} - \tfrac{2bq_2}{q_1^4} + \tfrac{c(q_1^2+4q_2^2)}{q_1^6}\,, \qquad a,b,c\in\mathbb{R}\,,\\
H_2 &= -\tfrac{p_1(p_1q_2-p_2q_1)}{2} - \tfrac{2aq_2}{q_1^2} + \tfrac{b(q_1^2+4q_2^2)}{q_1^4} - \tfrac{4cq_2(q_1^2+2q_2^2)}{q_1^6}\,.
\end{aligned}
\tag{35}
$$

According to [30] there is integrable deformation of this Hamilton function at $b=0$

$$
\bar{H}_1 = H + \Delta H = \frac{p_1^2+p_2^2}{4} + \frac{a}{q_1^2} + \frac{c(q_1^2+4q_2^2)}{q_1^6} + d(q_1^2+q_2^2) + \frac{e}{q_2^2}\,.
$$

The corresponding second integral of motion is the polynomial of fourth order in momenta. Our initial aim was to find variables of separation for this system in the framework of Abel theory. Instead of this, we find new integrable deformation of the Hamiltonian (35) with second polynomial integrals of sixth order in momenta. Below we describe construction of this new integrable system in details.

4.1 Separation of Variables and Abel Differential Equation

In parabolic coordinates Hamiltonians $H_{1,2}$ (35) look like

$$
\begin{aligned}
H_1 &= \tfrac{u_1p_{u_1}^2}{u_1-u_2} + \tfrac{u_2p_{u_2}^2}{u_2-u_1} - \tfrac{a}{u_1u_2} - \tfrac{b(u_1+u_2)}{u_1^2u_2^2} - \tfrac{c(u_1^2+u_1u_2+u_2^2)}{u_1^3u_2^3}\,,\\
H_2 &= \tfrac{u_1u_2p_{u_1}^2}{u_2-u_1} + \tfrac{u_1u_2p_{u_2}^2}{u_1-u_2} + \tfrac{a(u_1+u_2)}{u_1u_2} + \tfrac{b(u_1^2+u_1u_2+u_2^2)}{u_1^2u_2^2} + \tfrac{c(u_1+u_2)(u_1^2+u_2^2)}{u_1^3u_2^3}\,.
\end{aligned}
\tag{36}
$$

The corresponding Hamilton-Jacobi equations $H_{1,2}=\alpha_{1,2}$ are equivalent to separation relations

$$
\Phi(u_i,p_i,H_1,H_2) = \left(u_i^2p_{u_i}\right)^2 - (H_1u_i^4 + H_2u_i^3 - au_i^2 - bu_i - c) = 0\,,
$$

which determine the elliptic curve

$$
C: \qquad y^2 = f(x)\,, \qquad f(x) = \alpha_1u^4 + \alpha_2u^3 - au^2 - bu - c\,, \tag{37}
$$

i.e. variables

$$
x_{1,2} = u_{1,2}, \qquad y_{1,2} = u_{1,2}^2p_{u_{1,2}}
$$

may be identified with abscissas and ordinates of two points on C. In this case the number degrees of freedom $m=2$ does no equal to the genus $g=1$ of the underlying elliptic curve and, therefore, standard construction of BTs from [7,21] cannot be applied because it works only when $m=g$.

Equations of motion for $x_{1,2}$ are equal to

$$\frac{dx_1}{dt} = \{u_1, H_1\} = \frac{\partial H_1}{\partial p_{u_1}} = \frac{2y_1}{x_1(x_1-x_2)}, \qquad \frac{dx_2}{dt} = \{u_2, H_1\} = \frac{\partial H_1}{\partial p_{u_2}} = \frac{2y_2}{x_2(x_2-x_1)}. \tag{38}$$

It allows us to obtain Abel quadratures

$$\frac{x_1 dx_1}{y_1} + \frac{x_2 dx_2}{y_2} = 0, \qquad \text{and} \qquad \frac{x_1^2 dx_1}{y_1} + \frac{x_2^2 dx_2}{y_2} = 2dt,$$

which involve non-regular differentials [1]. Thus, following to Weierstrass idea [36], we change time $t \to s$ in (38) and introduce new equations

$$\frac{du_1}{ds} = \{u_1, H_1\}_W = \frac{2x_1y_1}{x_1-x_2}, \qquad \frac{du_2}{ds} = \{u_2, H_1\}_W = \frac{2x_2y_2}{x_2-x_1}. \tag{39}$$

in order to reduce Abel quadratures to the following form

$$\frac{dx_1}{x_1y_1} + \frac{dx_2}{x_2y_2} = 0, \qquad \frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 2ds_1. \tag{40}$$

After the Weierstrass change of time second quadrature incorporates standard holomorphic differential on the elliptic curve C that allows us to relate this equation with Picard group and Jacobian of C. Equations of motion (39) are Hamiltonian equations with respect to the new Poisson bracket

$$\{u_i, p_{u_j}\}_W = u_i^2\delta_{ij}, \qquad \{u_1, u_2\}_W = \{p_{u_1}, p_{u_2}\}_W = 0, \tag{41}$$

which is compatible with the original canonical bracket $\{.,.\}$ (15).

Suppose that transformation of variables

$$B: \qquad (u_1, u_2, p_{u_1}, p_{u_2}) \leftrightarrow (\widetilde{u}_1, \widetilde{u}_2, \widetilde{p}_{u_1}, \widetilde{p}_{u_2}) \tag{42}$$

preserves form of Hamilton equations (39) and Hamilton-Jacobi equations, i.e.

$$H_1(u, p_u) = \alpha_1 = H_1(\widetilde{u}, \widetilde{p}_{u_1}), \qquad H_2(u, p_u) = \alpha_2 = H_2(\widetilde{u}, \widetilde{p}_{u_1}),$$

where functions $H_{1,2}$ are given by (36). It means that variables $x_{1,2}, y_{1,2}$ and

$$x'_{1,2} = \widetilde{u}_{1,2}, \qquad y'_{1,2} = -\widetilde{u}^2_{1,2}\widetilde{p}_{u_{1,2}}.$$

satisfy to the Abel differential equation

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx'_1}{y'_1} + \frac{dx'_2}{y'_2} = 0. \tag{43}$$

In this case, points on the curve $P_{1,2} = (x_{1,2}y_{1,2})$ and $P'_{1,2} = (x'_{1,2}, y'_{1,2})$ form weight two unreduced divisors D and D' on C. An equivalence relations between these divisors we identify with desired auto-BT. Indeed, the corresponding Abel polynomial is equal to

$$\psi(x) = f(x) - \mathscr{P}(x)^2 = A(x-x_1)(x-x_2)(x-x'_1)(x-x'_2), \qquad A = \alpha_1 - b_2^2,$$

where b_2 is the coefficient of polynomial $\mathscr{P}(x) = b_2x^2 + b_1x + b_0$. It allows us to explicitly determine the desired mapping B (42) using polynomial

$$(x - x_1')(x - x_2') = \frac{f(x) - \mathscr{P}(x)^2}{A(x - x_1)(x - x_2)}\,, \tag{44}$$

where $f(x)$ is given by (37), and polynomial of the second order has the form

$$\mathscr{P}(x) = x\left(\frac{(x - x_2)y_1}{x_1(x_1 - x_2)} + \frac{(x - x_1)y_2}{x_2(x_2 - x_1)}\right). \tag{45}$$

Theorem 4. *If variables $\widetilde{u}_{1,2}$ are solutions of equation*

$$\left(H_1 - \left(\frac{u_1p_{u_1} - u_2p_{u_2}}{u_1 - u_2}\right)^2\right)x^2 - \frac{bu_1u_2 + c(u_1 + u_2)}{u_1^2u_2^2}\,x - \frac{c}{u_1u_2} = 0$$

and

$$\widetilde{p}_{u_{1,2}} = -\widetilde{u}_{1,2}^{-2}\,\mathscr{P}\big|_{x=\widetilde{u}_{1,2}}\,, \quad \mathscr{P}(x) = x\left(\frac{(x - u_2)u_1p_{u_1}}{u_1 - u_2} + \frac{(x - u_1)u_2p_{u_2}}{u_2 - u_1}\right),$$

then mapping B (42) preserves the form of Hamilton equations (39), the form of Hamiltonians $H_{1,2}$ (35) and (36) and the form of the Poisson bracket $\{.,.\}_W$ (41), i.e.

$$\{\widetilde{u}_i, \widetilde{p}_{u_j}\}_W = \widetilde{u}_i^2\delta_{ij}\,, \qquad \{\widetilde{u}_1, \widetilde{u}_2\}_W = \{\widetilde{p}_{u_1}, \widetilde{p}_{u_2}\}_W = 0\,.$$

The proof is a straightforward calculation.

Of course, we can not use this mapping B (42) for discretization of Hamiltonian flows (39), but this auto-BT is the hidden symmetry which describes fundamental properties of the given Hamiltonian system similar to the Noether symmetries.

4.2 Construction of the New Integrable System on $T^*\mathbb{R}$

The following Poisson map

$$\rho: \qquad (u_1, u_2, p_{u_1}, p_{u_2}) \to (u_1, u_2, u_1^2p_{u_1}, u_2^2p_{u_2})$$

reduces canonical Poisson bracket $\{.,.\}$ to bracket $\{.,.\}_W$, which allows us to rewrite equations of motion (39) in Hamiltonian form.

Using the composition of Poisson mappings ρ and B (42) we determine variables $\widehat{u}_{1,2}$, which are solutions of equation

$$\left(\rho(H_1) - \left(\frac{u_1^3p_{u_1} - u_2^3p_{u_2}}{u_1 - u_2}\right)^2\right)x^2 - \frac{bu_1u_2 + c(u_1 + u_2)}{u_1^2u_2^2}\,x - \frac{c}{u_1u_2} = 0 \tag{46}$$

where

$$\rho(H_1)=\frac{u_1^5p_{u_1}^2}{u_1-u_2}+\frac{u_2^5p_{u_2}^2}{u_2-u_1}-\frac{a}{u_1u_2}-\frac{b(u_1+u_2)}{u_1^2u_2^2}-\frac{c(u_1^2+u_1u_2+u_2^2)}{u_1^3u_2^3}\,.$$

The corresponding momenta are equal to

$$\widehat{p}_{u_{1,2}}=-\widehat{u}_{1,2}^{-4}\,\widehat{\mathscr{P}}\Big|_{x=\widehat{u}_{1,2}}\,,\qquad \widehat{\mathscr{P}}=x\left(\frac{(x-u_2)u_1^3p_{u_1}}{u_1-u_2}+\frac{(x-u_1)u_2^3p_{u_2}}{u_2-u_1}\right).\tag{47}$$

Straightforward calculation allows us to prove the following statement.

Theorem 5. *Canonical Poisson bracket*

$$\{u_i,p_{u_j}\}=\delta_{ij}\,,\qquad \{u_1,u_2\}=\{p_{u_1},p_{u_2}\}=0$$

has the same form

$$\{\widehat{u}_i,\widehat{p}_{u_j}\}=\delta_{ij}\,,\qquad \{\widehat{u}_1,\widehat{u}_2\}=\{\widehat{p}_{u_1},\widehat{p}_{u_2}\}=0$$

in variables $\widehat{u}_{1,2}$ and $\widehat{p}_{u_{1,2}}$ (46) and (47).

Thus, we obtain new canonical variables on phase space $T^*\mathbb{R}^2$, which can be useful to construction of new integrable systems in the frameworks of the Jacobi method.

For instance, let us substitute these variables into the separated relations similar to (33)

$$2\lambda_i=\left(\widehat{u}_i^4\widehat{p}_{u_i}^2+\frac{a}{\widehat{u}_i^2}+\frac{b}{\widehat{u}_i^3}+\frac{c}{\widehat{u}_i^4}\right)=\widehat{H}_1\pm\sqrt{\widehat{H}_2}\,,\qquad i=1,2,\tag{48}$$

and solve these relations with respect to $\widehat{H}_{1,2}$.

Theorem 6. *Functions on phase space $T^*\mathbb{R}^2$*

$$\widehat{H}_1=\lambda_1+\lambda_2\,,\qquad \widehat{H}_2=(\lambda_1-\lambda_2)^2\,,$$

are in involution with respect to the following compatible Poisson brackets

$$\{\widehat{u}_i,\widehat{p}_{u_j}\}=\delta_{i,j}\,,\qquad \{\widehat{u}_1,\widehat{u}_2\}=\{\widehat{p}_{u_1},\widehat{p}_{u_2}\}=0\,,\tag{49}$$

and

$$\{\widehat{u}_i,\widehat{p}_{u_j}\}'=\lambda_i^{-1}\delta_{i,j}\,,\qquad \{\widehat{u}_1,\widehat{u}_2\}'=\{\widehat{p}_{u_1},\widehat{p}_{u_2}\}'=0\,.\tag{50}$$

The proof is a straightforward calculation.

Moreover, using the corresponding bivectors $\mathbf{P}$ and $\mathbf{P}'$ it is easy to prove that vector field

$$X=\mathbf{P}d(\lambda_1+\lambda_2)=\mathbf{P}'d\left(\frac{\lambda_1^2+\lambda_2^2}{2}\right)$$

is bi-Hamiltonian vector field. This trivial in $\widehat{u}_{1,2}$ and $\widehat{p}_{u_{1,2}}$ variables Hamiltonian $\widehat{H}_1=\lambda_1+\lambda_2$ has more complicated form in original parabolic coordinates and momenta:

$$\widehat{H}_1=\frac{\big(bu_1u_2+c(3u_1+u_2)\big)u_1^4p_{u_1}^2}{c(u_1-u_2)}+\frac{\big(bu_1u_2+c(u_1+3u_2)\big)u_2^4p_{u_2}^2}{c(u_2-u_1)}-\frac{(bu_1+c)(au_1^2+bu_1+c)}{cu_1^4}$$
$$-\frac{(bu_2+c)(au_2^2+bu_2+c)}{cu_2^4}-\frac{4ac+b^2}{cu_1u_2}-\frac{5b(u_1+u_2)}{u_1^2u_2^2}-\frac{4c(u_1^2+u_1u_2+u_2^2)}{u_1^3u_2^3}\,.$$

Second integral is polynomial of sixth order in momenta

$$\widehat{H}_2=\left(cu_1^4u_2^4(u_1^2p_{u_1}-u_2^2p_{u_2})^2+\frac{(u_1-u_2)^2\Big((3u_1^2+2u_1u_2+3u_2^2)c^2+2u_1u_2\big(2au_1u_2+b(u_1+u_2)\big)c-b^2u_1^2u_2^2\Big)}{4}\right)$$
$$\times\left(\frac{u_1^4p_{u_1}^2-u_2^4p_{u_2}^2}{c(u_1-u_2)^2}-\frac{a(u_1+u_2)}{cu_1^2u_2^2(u_1-u_2)}-\frac{b(u_1^2+u_1u_2+u_2^2)}{cu_1^3u_2^3(u_1-u_2)}-\frac{(u_1+u_2)(u_1^2+u_2^2)}{u_1^4u_2^4(u_1-u_2)}\right)^2$$

If we put $a=b=0$ and then $c=0$, the second integral of motion $c\widehat{H}_2=K^2$ becomes a complete square. So, we have the geodesic flow on the plane with integrals of motion

$$T=\frac{u_1^4(3u_1+u_2)p_{u_1}^2}{u_1-u_2}+\frac{u_2^4(u_1+3u_2)p_{u_2}^2}{u_2-u_1}$$

and

$$K=\frac{u_1^2u_2^2(u_1^2p_{u_1}-u_2^2p_{u_2})(u_1^4p_{u_1}^2-u_2^4p_{u_2}^2)}{(u_1-u_2)^2}.$$

In Cartesian coordinates these integrals of motion read as

$$T=\frac{(q_1^2+6q_2^2)q_1^2p_1^2}{2}+(5q_1^2+12q_2^2)q_1q_2p_1p_2+\frac{(q_1^4+8q_1^2q_2^2+12q_2^4)p_2^2}{2}$$

and

$$K=\frac{q_1^7q_2p_1^3}{4}+\frac{q_1^6(q_1^2+6q_2^2)p_1^2p_2}{4}+q_1^5q_2(q_1^2+3q_2^2)p_1p_2^2+q_1^4(q_1^2+2q_2^2)q_2^2p_2^3\,.$$

It is easy to prove that Hamiltonian T has no polynomial integrals of motion of first or second order in momenta.

Such nonstandard Hamiltonians may appear in the study of a wide range of fields such as nonholonomic dynamics, control theory, seismology, biology, in the study of a self graviting stellar gas cloud, optoelectronics, fluid mechanics etc.

4.3 Other New Integrable Systems on the Plane

If we substitute parabolic coordinates $u_{1,2}$ and conjugated momenta $p_{u_{1,2}}$ into a family of separated relations

$$\begin{aligned}
A:&\ \big(u^2p_u\big)^2-(H_1u^4+H_2u^3+au^2+bu+c)=0\,,\\
B:&\ \big(u^2p_u\big)^2-(au^4+H_1u^3+H_2u^2+bu+c)=0\,,\\
C:&\ \big(u^2p_u\big)^2-(au^4+bu^3+H_1u^2+H_2u+c)=0\,,\\
D:&\ \big(u^2p_u\big)^2-(au^4+bu^3+cu^2+H_1u+H_2)=0
\end{aligned}\tag{51}$$

and solve the resulting pairs of equations with respect to $H_{1,2}$, we obtain dual Stäckel systems for which every trajectory of one system is a reparametrized trajectory of the other system [29]. The corresponding integrable diagonal metrics

$$\mathrm{g}_{km}=\begin{pmatrix}\frac{u_2^k u_1^m}{u_1-u_2} & 0\\ 0 & \frac{u_1^k u_2^m}{u_2-u_1}\end{pmatrix},\qquad k=0,1;\quad m=1,\dots,4,$$

are geodesically equivalent metrics [22].

For each of these systems we can construct an analogue of the Bäcklund transformation and Poisson map ρ, associated with the Weierstrass change of time, that allows us to get different canonical variables and different integrable systems on $T^*\mathbb{R}^2$. The first case in (51) was considered in the previous section, whereas the third case leads to an integrable system with quadratic integrals of motion. Thus, below we consider only the second and fourth separation relations in (51).

Case B. In this case

$$f(x)=ax^4+H_1x^3+H_2x^2+bx+c\,,\quad P(x)=x\left(\frac{(x-x_2)y_1}{x_1(x_1-x_2)}+\frac{(x-x_1)y_2}{x_2(x_2-x_1)}\right)$$

and variables $x_{3,4}=\widetilde{u}_{1,2}$ are the roots of polynomial

$$\frac{\psi(x)}{(x-u_1)(x-x_2)}=\left(a-\frac{(u_1p_{u_1}-u_2p_{u_2})^2}{(u_1-u_2)^2}\right)x^2+\frac{bu_1u_2+c(u_1+u_2)}{u_1^2u_2^2}\,x+\frac{c}{u_1u_2}\,.$$

These coordinates commute with respect to the Poisson brackets

$$\{u_i,p_{u_j}\}_W=u_i\delta_{ij}\,,\qquad \{u_1,u_2\}_W=\{p_{u_1},p_{u_2}\}_W=0\,.$$

Using an additional Poisson map

$$\rho_B:\qquad (u_1,u_2,p_{u_1},p_{u_2})\to(u_1,u_2,u_1p_{u_1},u_2p_{u_2})\,,$$

we can define canonical variables $\widehat{u}_{1,2}$ on $T^*\mathbb{R}^2$, which are the roots of polynomial

$$\left(a-\frac{(u_1^2p_{u_1}-u_2^2p_{u_2})^2}{(u_1-u_2)^2}\right)x^2+\frac{bu_1u_2+c(u_1+u_2)}{u_1^2u_2^2}\,x+\frac{c}{u_1u_2}\,,$$

and the conjugated momenta

$$\widehat{p}_{u_{1,2}}=-\widehat{u}_{1,2}^{-3}\,\widehat{\mathscr{P}}\Big|_{x=\widehat{u}_{1,2}}\,,\qquad \widehat{\mathscr{P}}=x\left(\frac{(x-u_2)u_1^2p_{u_1}}{u_1-u_2}+\frac{(x-u_1)u_2^2p_{u_2}}{u_2-u_1}\right)\,,$$

so that

$$\{\widehat{u}_i,\widehat{p}_{u_j}\}=\delta_{ij}\,,\qquad \{\widehat{u}_1,\widehat{u}_2\}=\{\widehat{p}_{u_1},\widehat{p}_{u_2}\}=0\,.$$

Substituting these canonical variables into the separated relations

$$2\lambda_i=\left(\widehat{u}_i^3\widehat{p}_{u_i}^2-a\widehat{u}_i-\frac{b}{\widehat{u}_i^2}-\frac{c}{\widehat{u}_i^3}\right)=\widehat{H}_1\pm\sqrt{\widehat{H}_2}\,,\quad i=1,2,$$

one gets Hamilton function

$$\widehat{H}_1 = \frac{u_1^3\Big(bu_1u_2+c(3u_1+u_2)\Big)p_{u_1}^2}{c(u_1-u_2)} + \frac{u_2^3\Big(bu_1u_2+c(u_1+3u_2)\Big)p_{u_2}^2}{c(u_2-u_1)} - a\left(\frac{bu_1u_2}{c} + 3(u_1+u_2)\right)$$

$$+\frac{b^2(u_1+u_2)}{cu_1u_2} + \frac{b(u_1+2u_2)(2u_1+u_2)}{u_1^2u_2^2} + \frac{c(u_1+u_2)(u_1^2+3u_1u_2+u_2^2)}{u_1^3u_2^3}$$

and the second integral of motion, which is polynomial of sixth order in momenta

$$\widehat{H}_2 = \left(\frac{u_1^3u_2^3(u_1^2p_{u_1}-u_2^2p_{u_2})^2}{c(u_1-u_2)^2} - \frac{au_1^3u_2^3}{c} + \frac{b^2u_1^2u_2^2}{4c^2} + \frac{b(u_1+u_2)u_1u_2}{2c} + \frac{(u_1+u_2)^2}{4}\right)$$

$$\times\frac{4}{(u_1-u_2)^2}\left(a(u_1-u_2) + b\left(\frac{1}{u_1^2} - \frac{1}{u_2^2}\right) + c\left(\frac{1}{u_1^3} - \frac{1}{u_2^3}\right) - u_1^3p_{u_1}^2 + u_2^3p_{u_2}^2\right)^2 .$$

Vector field associated with Hamiltonian $\widehat{H}_1$ is a bi-Hamiltonian vector field with respect to the compatible Poisson brackets (49) and (50).

If we put $a = b = 0$ and then $c = 0$, we obtain a geodesic flow on the plane with an integral of motion of third order in momenta

$$\begin{aligned} \widehat{H}_1 = T &= \frac{u_1^3(3u_1+u_2)p_{u_1}^2}{u_1-u_2} + \frac{u_2^3(u_1+3u_2)p_{u_2}^2}{u_2-u_1}\,, \\ K &= \frac{u_1^{3/2}u_2^{3/2}(u_1^2p_{u_1}-u_2^2p_{u_2})(u_1^3p_{u_1}^2-u_2^3p_{u_2}^2)}{(u_1-u_2)^2}\,. \end{aligned} \tag{52}$$

In original Cartesian coordinates these integrals of motion have the form

$$\widehat{H}_1 = T = \frac{3q_1^2q_2}{2}p_1^2 + (q_1^2+6q_2^2)q_1p_1p_2 + \frac{q_2(5q_1^2+12q_2^2)}{2}\,p_2^2$$

and

$$K = q_1^6p_1^3 + 6q_1^5q_2p_1^2p_2 + q_1^4(q_1^2+12q_2^2)p_1p_2^2 + 2(q_1^2+4q_2^2)q_1^3q_2p_2^3\,.$$

Construction and classification of all the integrable geodesic flows on Riemannian manifolds is a classical problem in Riemannian geometry [3,19,22]. We propose to use auto-BTs to solution of this problem.

Case D. In this case

$$f(x) = ax^4 + bx^3 + cx^2 + H_1x + H_2\,, \quad P(x) = x\left(\frac{(x-x_2)y_1}{x_1(x_1-x_2)} + \frac{(x-x_1)y_2}{x_2(x_2-x_1)}\right)$$

and variables $x_{3,4} = \widetilde{u}_{1,2}$ are the roots of polynomial

$$\frac{\psi(x)}{(x-u_1)(x-x_2)} = \left(a - \frac{(u_1p_{u_1}-u_2p_{u_2})^2}{(u_1-u_2)^2}\right)x^2 + \left(a(u_1+u_2) + b - \frac{u_1^2p_{u_1}^2-u_2^2p_{u_2}^2}{u_1-u_2}\right)x$$

$$+\left(a(u_1^2+u_1u_2+u_2^2) + b(u_1+u_2) + c - \frac{u_1^3p_{u_1}^2-u_2^3p_{u_2}^2}{u_1-u_2}\right).$$

These coordinates commute with respect to the Poisson brackets

$$\{u_i, p_{u_j}\}_W = u_i^{-1}\delta_{ij}\,, \qquad \{u_1, u_2\}_W = \{p_{u_1}, p_{u_2}\}_W = 0$$

associated with the Weierstrass change of time.

Using an additional Poisson map

$$\rho_D: \qquad (u_1, u_2, p_{u_1}, p_{u_2}) \to (u_1, u_2, u_1^{-1}p_{u_1}, u_2^{-1}p_{u_2})\,,$$

we can define canonical variables $\widehat{u}_{1,2}$ on $T^*\mathbb{R}^3$, which are the roots of polynomial

$$\rho_D\left(\tfrac{\psi(x)}{(x-u_1)(x-x_2)}\right) = \left(a - \tfrac{(p_{u_1}-p_{u_2})^2}{(u_1-u_2)^2}\right)x^2 + \left(a(u_1+u_2) + b - \tfrac{p_{u_1}^2-p_{u_2}^2}{u_1-u_2}\right)x$$

$$+\left(a(u_1^2+u_1u_2+u_2^2) + b(u_1+u_2) + c - \tfrac{u_1p_{u_1}^2-u_2p_{u_2}^2}{u_1-u_2}\right)\,,$$

and the conjugated momenta

$$\widehat{p}_{u_{1,2}} = -\widehat{u}_{1,2}^{-1}\,\widehat{\mathscr{P}}\Big|_{x=\widehat{u}_{1,2}}\,, \qquad \widehat{\mathscr{P}} = x\left(\frac{(x-u_2)p_{u_1}}{u_1-u_2} + \frac{(x-u_1)p_{u_2}}{u_2-u_1}\right)\,,$$

so that

$$\{\widehat{u}_i, \widehat{p}_{u_j}\} = \delta_{ij}\,, \qquad \{\widehat{u}_1, \widehat{u}_2\} = \{\widehat{p}_{u_1}, \widehat{p}_{u_2}\} = 0\,.$$

Substituting these variables into the separation relations

$$2\lambda_i = \left(\widehat{u}_i\widehat{p}_{u_i}^2 - a\widehat{u}_i^3 - b\widehat{u}_i^2 - c\widehat{u}_i\right) = \widehat{H}_1 \pm \sqrt{\widehat{H}_2}\,, \quad i = 1, 2,$$

one gets Hamilton function

$$\widehat{H}_1 = \frac{u_1(2u_1+u_2)p_{u_1}^2}{u_1-u_2} + \frac{u_2(u_1+2u_2)p_{u_2}^2}{u_2-u_1} - a(u_1+u_2)(2u_1^2+u_1u_2+2u_2^2)$$
$$- b(2u_1^2+3u_1u_2+2u_2^2) - 2c(u_1+u_2) \tag{53}$$

and the second integral of motion, which is polynomial of fourth order in momenta

$$\widehat{H}_2 = u_1^2u_2^2\Big(\tfrac{(p_{u_1}-p_{u_2})^3\left((3u_1+u_2)p_{u_1}+(u_1+3u_2)p_{u_2}\right)}{(u_1-u_2)^3} - \tfrac{4c(p_{u_1}-p_{u_2})^2}{(u_1-u_2)^2}$$

$$-\tfrac{2a\left(3(u_1^2+u_2^2)(p_{u_1}^2+p_{u_2}^2)-4(u_1^2+u_1u_2+u_2^2)p_{u_1}p_{u_2}\right)}{(u_1-u_2)^2} - \tfrac{2b(p_{u_1}-p_{u_2})\left((u_1+3u_2)p_{u_1}-(3u_1+u_2)p_{u_2}\right)}{(u_1-u_2)^2}$$

$$+a^2(3u_1^2+2u_1u_2+3u_2^2) + a(2b(u_1+u_2)+4c) - b^2\Big)\,.$$

Vector field associated with Hamiltonian $\widehat{H}_1$ is a bi-Hamiltonian vector field with respect to the compatible Poisson brackets (49) and (50).

If we put $a = b = 0$ and then $c = 0$, we obtain a geodesic flow with an integral of motion, which is polynomial of fourth order in momenta

$$\widehat{H}_1 = T = \frac{u_1(2u_1+u_2)p_{u_1}^2}{u_1-u_2} + \frac{u_2(u_1+2u_2)p_{u_2}^2}{u_2-u_1}\,,$$
$$K = \frac{u_1^2u_2^2(p_{u_1}-p_{u_2})^3\Big((3u_1+u_2)p_{u_1}+(u_1+3u_2)p_{u_2}\Big)}{(u_1-u_2)^3}\,.$$

This integrable system is superintegrable system with one more integral of motion, which is a polynomial of third order in momenta

$$J = \frac{u_1u_2(p_{u_1}-p_{u_2})^2(u_1^2p_{u_1}-u_2^2p_{u_2})}{(u_1-u_2)^3}\,,$$

so that

$$\{T,K\}=0\,,\qquad \{T,J\}=0\,,\qquad \{J,K\}\neq 0\,.$$

In original Cartesian coordinates these integrals of motion are

$$T = q_2\left(\frac{p_1^2}{2}+p_2^2\right)+\frac{q_1p_1p_2}{2}\,,\qquad K=\frac{(q_1^2-q_2^2)p_1^4}{4}+\frac{q_1q_2p_1^3p_2}{2}$$

and

$$J = (q_1p_1+2q_2p_2)p_1^2\,.$$

Using additional canonical transformation we can reduce polynomial T to the following form

$$T = m_1(q_1,q_2)p_1^2 + m_2(q_1,q_2)p_2^2$$

and obtain new example of superintegrable system with the position dependent masses.

5 Conclusion

We prove that the auto-Bäcklund transformation of Hamilton-Jacobi equation which represents symmetry of the level manifolds can be also useful in classical mechanics. In particular, we prove that geodesic flow associated with Hamiltonian

$$T = \sum_{i,j=1}^{2} \mathrm{g}_{ij}^{(km)}(u)p_{u_i}p_{u_j}\,,$$

where

$$\mathrm{g}^{(km)} = \begin{pmatrix} \frac{(ku_1+u_2)u_1^m}{u_1-u_2} & 0 \\ 0 & \frac{(ku_2+u_1)u_2^m}{u_2-u_1}\end{pmatrix} \tag{54}$$

is integrable for $k=3$; $m=3,4$ and for $k=2$; $m=1$. The corresponding integrals of motion

$$K = (u_1^2p_{u_1}-u_2^2p_{u_2})\cdot\frac{u_1^{m/2}u_2^{m/2}(u_1^mp_{u_1}^2-u_2^mp_{u_2}^2)}{(u_1-u_2)^2}\,,\qquad m=3,4$$

and

$$K=(u_1^2p_{u_1}-u_2^2p_{u_2})\cdot\frac{u_1^mu_2^m(p_{u_1}-p_{u_2})^2}{(u_1-u_2)^3}\,,\qquad m=1$$

are polynomials of the third order in momenta with a common factor. It allows us to suppose, that there are similar integrable systems for other values of k and m in (54), see discussion in [3,19,22]. Indeed, substituting

$$K=(u_1^2p_{u_1}-u_2^2p_{u_2})\cdot\big(f(u_1,u_2)p_{u_1}^2+g(u_1,u_2)p_{u_1}p_{u_2}+h(u_1,u_2)p_{u_2}^2\big)$$

into the equation $\{T,K\}=0$ and solving the resulting system of PDE's with respect to functions f,g and h we can prove the following proposition.

Theorem 7. *Metric* $\mathrm{g}^{(km)}$ *(54) yields integrable geodesic flow on the plane at* $m=1,k=2$ *and*

$$m=3\,,\qquad k=\pm1,3,\frac{1}{2}\,,\qquad \textit{and}\qquad m=4\,,\qquad k=\pm1,\pm3,-\frac{3}{5},-\frac{1}{7},\frac{1}{5},\frac{1}{2}\,.$$

In the similar manner we can study common properties of the obtained variables of separation, compatible Poisson brackets and recursion operators. It can be useful for investigation of other integrable systems with integrals of motion of third, fourth and sixth order in momenta. In particular we hope to use obtained experience for the study of Toda lattice associated with G_2 root system.

Starting with other well-known separable Hamilton-Jacobi equations we can also obtain new integrable systems with polynomial integrals of motion. For instance, we can take Hénon-Heiles system on the sphere [2] or nonholonomic Heisenberg type system with quadratic integrals of motion from [11] and obtain new integrable systems with quartic integrals of motion. We plan to describe some of these systems in the forthcoming publications.

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Quadro-Cubic Cremona Transformations and Feigin-Odesskii-Sklyanin Algebras with 5 Generators

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Abstract. We study different algebraic and geometric properties of Heisenberg (H-) invariant Poisson polynomial algebras with 5 generators. These algebras are unimodular, and the elliptic Feigin-Odesskii-Sklyanin Poisson algebras $q_{n,k}(Y)$ constitute the main important example. We discuss all the quadratic H-invariant Poisson tensors on $\mathbb{C}^5$. For the Sklyanin algebras $q_{5,1}(Y)$ and $q_{5;2}(Y)$, we explicitly write the Poisson morphisms on the moduli space of the vector bundles on the normal elliptic curve Y in $\mathbb{P}^4$, studied by Polishchuk and Odesskii-Feigin as the quadro-cubic Cremona transformation on $\mathbb{P}^4$.

Keywords: Cremona transformations
Feigin-Odesskii-Sklyanin algebras · Sklyanin elliptic algebras

1 Introduction

The elliptic or Sklyanin algebras are one of the most studied and important examples of the so-called *algebras with quadratic relations*. This class of algebras possesses many wonderful and useful properties. It is well-known (see, for example, [9] or [25]) that they can be defined as a quotient $T(V)/(R)$ of the tensor algebra $T(V)$ of an n-dimensional vector space V over a space R of quadratic relations. (Practically, V is identified with sections of a degree n vector bundle over a fixed elliptic curve Y. This also explains the name "elliptic algebra"). These are classes of Noetherian graded associative algebras which are Koszul, Cohen-Macaulay and have the Hilbert function as a polynomial ring with n variables. The first $n = 4$ examples of such algebras were discovered by E. Sklyanin in his studies of integrable discrete and continuous Landau-Lifshitz models by the Quantum Inverse Scattering Method [34,35]. These algebras are intensively studied [10,36–38]. Elliptic algebras with 3 generators were discovered by M. Artin,

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W. Schelter and J. Tate. Later (due to P. Smith, T. Stafford, their students and collaborators among which we should mention M. Van den Bergh), the cases of $n = 3, 4$ were studied in great details [1,40].

A systematic description of the Sklyanin elliptic algebras of any Gelfand-Kirillov dimension $n \geq 3$ and their far generalisations were proposed by B. Feigin and A. Odesskii in their preprints and papers of 1980–90. Their approach was based upon some deformation quantisation related to the addition law on the underlying elliptic curve. Here, we shall quote some of their articles which are relevant to our aims (see the review paper [25] and full list of the references there).

In this chapter, we shall focus on the important invariance property of the Sklyanin elliptic algebras. Let us recall that if we have an n-dimensional vector space V and fix a base $v_0, \dots, v_{n-1}$ of V, then the *Heisenberg group of dimension n in the Schrödinger representation* is the subgroup $H_n \subset GL(V)$ generated by the operators

$$\sigma : v_i \to v_{i-1}; \tau : vi \to \varepsilon_i(v_i); (\varepsilon_i)^n = 1; 0 \leq i \leq n-1.$$

This group is of order n^3 and is a central extension

$$1 \to \mathbb{U}_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 1,$$

where $\mathbb{U}_n$ is the group of n-th roots of unity. This action provides the automorphisms of the Sklyanin algebra, which are compatible with the grading, and also defines an action on the "quasi-classical" limit of the Sklyanin algebras, namely the elliptic quadratic Poisson structures on $\mathbb{P}^{n-1}$. The latter are identified with Poisson structures on some moduli spaces of the degree n and rank $k+1$ vector bundles with parabolic structure (= the flag $0 \subset F \subset \mathbb{C}^{k+1}$ on an elliptic curve Y). They denoted these elliptic Poisson algebras by $q_{n,k}(Y)$. The algebras $q_{n,k}(Y)$ arise in the Feigin-Odesskii "deformational" approach and form a subclass of *quadratic polynomial Poisson structures*.

One of the aims of this chapter is to continue our studies of the polynomial Poisson algebras, which were started in [23,27,28], and, in particular, to describe explicitly the five-dimensional Heisenberg-invariant *quadratic* polynomial Poisson tensors corresponding to $q_{5,1}(Y)$ and $q_{5,2}(Y)$. Our results for $n = 5$ complete some unpublished computations of A. Odesskii and shed light on the geometry behind the elliptic Poisson algebras $q_{5,1}(Y)$ and $q_{5,2}(Y)$.

These algebras are $\eta \to 0$ limits of the "quantum" Sklyanin algebras $Q_{5,1}(Y;\eta)$ and $Q_{5,2}(Y;\eta)$, which were described in the famous "Kiev preprint" of Feigin and Odesskii [9]. In particular, they showed that the algebra $Q_{5,1}(Y;\eta)$ embeds as a subalgebra into the algebra $Q_{5,2}(Y;\eta)$ and vice versa. We argue that these embedding homomorphisms are the "quantum" analogues of the classical quadro-cubic Cremona transformations of $\mathbb{P}^4$ [32], and verify that they are Poisson birational morphisms that presumably correspond to compositions of Polishchuk birational mappings between moduli spaces $\mathscr{M}od(E_1, E_2, f)$ of stable triples of vector bundles and their morphisms on the given elliptic curve [30].

The Cremona transformations play an important role in our description of the symplectic foliations of the structures $q_{5,1}(Y)$ and $q_{5,2}(Y)$. Symplectic leaves of $q_{5,1}(Y)$ have dimension $0, 2$ and 4. A two-dimensional symplectic leaf M^1 (in the notations of [9] and Subsection 2.2.4) is the cone $C_1(Y)$ of the curve Y. Note that our elliptic curve Y can be identified with the 1-st *secant variety* $\mathrm{Sec}_1(\mathrm{Y})$ of Y (see precise definition below in 2.3.4). The four-dimensional leaves M^2 were described by Feigin and Odesskii as the cone $C_2(Y) := C(\mathrm{Sec}(\mathrm{Y}))$ over the unification of all chords of Y, or, in other words, the cone over the 2-nd secant variety $\mathrm{Sec}_2(\mathrm{Y}) := \mathrm{Sec}(\mathrm{Y})$, and these are the level hypersurfaces of the unique *Casimir* $K(C_2(Y))$, which is a polynomial of degree 5(the center $Z(q_{5,1}(Y)) = \mathbb{C}[K(C_2(Y))]$),

$$M^2 = \{K(C_2(Y)) = 0\} \subset \mathbb{C}^5.$$

Thus, we obtain that the inclusion of the symplectic leaves

$$M^0 \subset M^1 \subset M^2$$

corresponds to the inclusions of secant varieties of Y:

$$\{0\} = \mathrm{Sec}_0(\mathrm{Y}) \subset \mathrm{Y} \simeq \mathrm{Sec}_1(\mathrm{Y}) \subset \mathrm{Sec}(\mathrm{Y}) \simeq \mathrm{Sec}_2(\mathrm{Y}).$$

Here, we fix the origin $0 \in Y$. We shall extract the explicit determinant representation for the symplectic leaves M^2 from the projective geometry of the elliptic curve [16].

The geometry of $q_{5,2}(Y)$ is also very interesting. Feigin and Odesskii in [9] had observed that the union of two-dimensional leaves for $q_{5;2}(Y)$ is the cone $C(X)$ in $\mathbb{C}^5$ over two-dimensional surface $X := S^2(Y)$ which is embedded in $\mathbb{P}^4$. The description of four-dimensional leaves is based on the notion of a *trisecant variety* $\mathrm{Trisec}(X)$ for the elliptic scroll $X = S^2(Y)$. This variety is a union of all the lines in

$$X = S^2(Y) = Y \times Y/\mathbb{Z}_2$$

passing through pairs $(\xi;\xi_1)$ and $(\xi;\xi_2)$ in X. For fixed $\xi \in Y$, if $Y_\xi \subset X$ is a curve formed by of points $(\xi;\xi_1)$, then it can be embedded in the projective plane $\mathbb{P}^2_\xi \subset \mathbb{P}^4$.

The four-dimensional leaves are cones $C(\mathrm{Trisec}(X))$ that are in fact the cones over $\sqcup_\xi \mathbb{P}^2_\xi$. The variety $\mathrm{Trisec}(X)$ is a quintic hypersurface in $\mathbb{P}^4$, and, therefore, there is a quintic polynomial $K_{5,2}(\mathrm{Trisec}(X))$, such that the leaves are level hypersurfaces similarly to the case of $q_{5,1}(Y)$. We shall give this polynomial explicitly and explain its projective geometric meaning as the Jacobian determinant of the inverse Cremona transformation (we remind the definitions below in Sect. 4.1).

It is well-known, since the works of Bianchi [6] and F. Klein lectures on the icosahedron [18], that the normal elliptic curves of degree $n \geq 4$ in $\mathbb{P}^{n-1}$ are given by $l = \frac{n(n-3)}{2}$ quadrics $q_1, \ldots, q_l$. More precisely, it means that the projective coordinate ring of the curve is the quotient $\mathbb{C}[Y] = \mathbb{C}[\mathbb{P}^{n-1}]/\mathscr{I}(q_1, \ldots, q_l)$ of the polynomial ring $\mathbb{C}[x_0, \dots, x_{n-1}]$ by the ideal generated by l quadrics. It is worth to remark that, here, we mean the schematic intersection only and, reciprocally, it is not true in general that *any* l quadrics define an elliptic curve. However, it is true if this quadrics are *Pfaffians* of some linear form matrix. In particular, for $n = 5$, the number $l = 5$, and we can take the following Heisenberg invariant family of quadric generators:

$$q_i = {x_i}^2 + a x_{i+2} x_{i+3} - \frac{x_{i+1} x_{i+4}}{a}, \quad i \in \mathbb{Z}_5, \quad a \in \mathbb{C}^*. \tag{1}$$

We shall remind this Klein description in Sect. 4. Note that, in the case $n \geq 5$, the homogeneous ideal is not a complete intersection generated by 4×4 Pfaffians of 5×5 alternating (usually refer as Klein) matrix. These Pfaffians are exactly the quadrics q_i, $0 \leq i \leq 4$.

In the language of classical projective geometry, the Pfaffians q_i define a quadratic Cremona transformation of $\mathbb{P}^4$ [33]. The inverse Cremona transformation is defined by the homogeneous ideal of a 3-fold which is the secant variety of the initial curve generating by five cubics:

$$\begin{aligned} \mathscr{Q}_i(z) = z_i^3 + k(z_{i+1}^2 z_{i+3} + z_{i+2} z_{i+4}^2) - k^{-1}(z_{i+1} z_{i+2}^2 + z_{i+3}^2 z_{i+4}) - \quad (2) \\ -k^2 z_i z_{i+1} z_{i+4} - k^{-2} z_i z_{i+2} z_{i+3}, \qquad i \in \mathbb{Z}_5, \qquad k \in \mathbb{C}^*. \end{aligned}$$

The quintic polynomial $K_{5,1}(C_2(Y))$ defining the 4-dimensional symplectic leaf M^2 is the determinant of the Jacobi matrix

$$K_{5,1}(C_2(Y))(x, a) = \det \frac{\partial(q_0, q_1, q_2, q_3, q_4)}{\partial(x_0, x_1, x_2, x_3, x_4)},$$

while the second Casimir $K_{5,2}(C(\mathrm{Trisec} X))(z, k)$ is the determinant of some *syzygy* 5×5-matrix $L_k(z)$ and it defines symplectic leaves of the algebra $q_{5,2}(Y)$.

We shall give the explicit values for both Casimirs and for $L_k(z)$ in Sects. 3–5.

One should remark that there also exists the Jacobian representation

$$K_{5,2}(C(\mathrm{Trisec}(X))(z, k) = \det \frac{\partial(\mathscr{Q}_0, \mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \mathscr{Q}_4)}{\partial(z_0, z_1, z_2, z_3, z_4)},$$

where the polynomials $\mathscr{Q}_j, j = 0, \dots, 4$ play the role of quadrics - Pfaffians of some other Klein matrix defining some other Heisenberg -invariant degree 15 elliptic curve $\tilde{Y}$ (which is isomorphic to Y as the image of the latter via the *Horrocks-Mumford map*). This result is a corollary of the geometric description in [4].

The chain of transformations

$$\mathbb{C}[x_0, x_1, x_2, x_3, x_4] \mapsto \mathbb{C}[z_0, z_1, z_2, z_3, z_4] \mapsto \mathbb{C}[x_0, x_1, x_2, x_3, x_4],$$

where $z_i = q_i(x)$ and, then, $x_i = \mathscr{Q}_i(z)$ reduces to the multiplication

$$x_i \mapsto K_{5,1}(C_2(Y))(x, a) x_i,$$

for each $0 \leq i \leq 4$. In the same way, the transformations

$$\mathbb{C}[z_0, z_1, z_2, z_3, z_4] \mapsto \mathbb{C}[x_0, x_1, x_2, x_3, x_4] \mapsto \mathbb{C}[z_0, z_1, z_2, z_3, z_4]$$

with $x_i = \mathscr{Q}_i(z)$ and then $z_i = q_i(x)$ are reduced to the multiplication

$$z_i \mapsto K_{5,2}(C(X))(z,k)z_i,$$

for each $0 \leq i \leq 4$.

It is obvious that the Poisson brackets in $\mathbb{C}^5(x)$ are "conformally" preserved by the chains:

$$\{K_{5,1}(C_2(Y))x_i, K_{5,1}(C_2(Y))x_j\}_{5,1} = K_{5,1}(C_2(Y))^2\{x_i, x_j\}_{5,1},$$

as well as their analogues in $\mathbb{C}[z_0, z_1, z_2, z_3, z_4]$.

This fact presumably reflects the property $\phi^3 = 1$ of Polishchuk's birational morphisms of moduli spaces of triples in the case of Odesskii-Feigin triple moduli spaces.

Is much less evident that the intermediate Cremona transformations are also birational Poisson morphisms for these different elliptic Poisson structures. We shall show it by direct verification. Finally, we stress that the Heisenberg invariance condition describes all the 14 solutions of the classification problem for quadratic homogeneous Poisson structures in $d = 5$. Namely, we obtain two solutions which correspond to the Feigin-Odesskii-Sklyanin algebras $q_{5,1}(Y)$ and $q_{5,2}(Y)$, and, apart from them, twelve exceptional quadratic Poisson structures defined by vertices of the Klein icosahedron on the sphere $\mathbb{P}^1$ (which is nothing but the rational modular curve $X(5)$ in the Horrocks-Mumford geometric picture).

This chapter has two-fold aim. First, as it was mentioned earlier, we would like to give a few "geometric" comments on the results of Odesskii and Feigin [9] and to draw attention to the intimate link between the Cremona transformation geometry in $d = 5$ and different families of elliptic algebras. Second, and this is an important motivation, this chapter is an extended written version of the author's lectures at the Laboratory of *Algebraic Geometry and its Applications*, of the Moscow Higher School of Economics, and at the International Conference of Mathematical Physics *Kezenoy-Am 2016*. Thus, we include in the chapter many reminders and a review of old results obtained in our previous works with Odesskii [22, 23] and Ortenzi et al. [27].

The chapter is organised as follows: Section 2 covers some known facts about polynomial quadratic and elliptic (Poisson) algebras. This section is strongly based on our papers [22, 23]. Then, in Sect. 3, we review the quadratic H-invariant Poisson tensor with 5 independent variables. The main focus here is on the Feigin-Odesskii-Sklyanin elliptic Poisson algebras $q_{5,1}(Y)$ and $q_{5,2}(Y)$. We describe the exact solutions for quadratic H-invariant Poisson tensors, and we show that they correspond to two generic cases, which are associated with smooth degree 5 elliptic curve configurations, and to 12 exceptional ones which correspond to Odesskii rational degenerations associated with pentagonal configurations of rational curves [24].

Sections 4 and 5 describe the relations between the classical quadro-cubic Cremona transformations in $\mathbb{P}^4$ (via the elements of the Horrocks-Mumford bundle geometry) and these 14 classes of quadratic Poisson structures in $\mathbb{P}^4$.

2 Poisson Algebras and Elliptic Poisson Algebras

2.1 Poisson Algebras on Affine Varieties

By polynomial Poisson structures we understand those ones whose brackets are polynomial in terms of local coordinates on the underlying Poisson manifold. A typical example of a such structure is the famous Sklyanin algebra.

Recall that a Poisson structure on a manifold M (it does not play an important role whether it is smooth or algebraic) is given by a bivector antisymmetric field $\pi \in \Lambda^2(TM)$ defining on the corresponding algebra of functions on M a structure of (infinite dimensional) Lie algebra by means of the Poisson brackets

$$\{f, g\} = \langle \pi, df \wedge dg \rangle.$$

The Jacobi identity for this bracket is equivalent to an analogue of the (classical) Yang-Baxter equation, namely to the "Poisson Master Equation": $[\pi, \pi] = 0$, where the brackets $[,] : \Lambda^p(TM) \times \Lambda^q(TM) \mapsto \Lambda^{p+q-1}(TM)$ are the only Lie super-algebra structure on $\Lambda^{\cdot}(TM)$ given by the so-called Schouten brackets. For all these facts, we refer, for example, to an excellent recent survey [31].

Let us consider $n-2$ polynomials $\mathscr{Q}_i$ in $\mathbb{C}^n$ with coordinates x_i, $i = 1, \dots, n$. For any polynomial $\lambda \in \mathbb{C}[x_1, \dots, x_n]$, we can define a bilinear differential operation

$$\{,\} : \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[x_1, \dots, x_n] \mapsto \mathbb{C}[x_1, \dots, x_n]$$

by the formula

$$\{f, g\} = \lambda \frac{df \wedge dg \wedge d\mathscr{Q}_1 \wedge \dots \wedge d\mathscr{Q}_{n-2}}{dx_l \wedge dx_2 \wedge \dots \wedge dx_n}, \; f, g \in \mathbb{C}[x_1, \dots, x_n]. \tag{3}$$

This operation gives a Poisson algebra structure on $\mathbb{C}[x_1, \dots, x_n]$ as a partial case of a more general $n-m$-ary Nambu operation given by an antisymmetric $n-m$-polyvector field η:

$$\langle \eta, df_1 \wedge \dots \wedge df_{n-m} \rangle = \big\{f_1, \dots, f_{n-m}\big\},$$

depending on m polynomial "Casimirs" $\mathscr{Q}_1, \dots, \mathscr{Q}_m$ and λ such that

$$\{f_1, \dots, f_{n-m}\} = \lambda \frac{df_1 \wedge \dots \wedge df_{n-m} \wedge d\mathscr{Q}_1 \wedge \dots \wedge d\mathscr{Q}_m}{dx_l \wedge dx_2 \wedge \dots \wedge dx_n}, \; f_i \in \mathbb{C}[x_1, \dots, x_n], \tag{4}$$

and

$$\big\{, \dots, \big\} : \mathbb{C}[x_1, \dots, x_n]^{\otimes n-m} \mapsto \mathbb{C}[x_1, \dots, x_n]$$

so that the three following properties hold:

(1) antisymmetricity:

$$\{f_1,\ldots,f_{n-m}\} = (-1)^{\sigma}\{f_{\sigma(1)},\ldots,f_{\sigma(n-m)}\}, \sigma \in Symm_{n-m};$$

(2) coordinate-wise "Leibnitz rule" for any $h \in \mathbb{C}[x_1,\ldots,x_n]$:

$$\{f_1 h,\ldots,f_{n-m}\} = f_1\{h,\ldots,f_{n-m}\} + h\{f_1,\ldots,f_{n-m}\};$$

(3) the "Fundamental Identity" (which replaces the Jacobi):

$$\begin{gathered}\{\{f_1,\ldots,f_{n-m}\}, f_{n-m+1},\ldots,f_{2(n-m)-1}\}+\\ \{f_{n-m},\{f_1,\ldots,(f_{n-m})^{\vee} f_{n-m+1}\}, f_{n-m+2},\ldots,f_{2(n-m)-1}\}+\\ +\{f_{n-m},\ldots,f_{2(n-m)-2},\{f_1,\ldots,f_{n-m-1},f_{2(n-m)-1}\}\} =\\ \{f_1,\ldots,f_{n-m-1},\{f_{n-m},\ldots,f_{2(n-m)-1}\}\}\end{gathered}$$

for any $f_1,\ldots,f_{2(n-m)-1} \in \mathbb{C}[x_1,\ldots,x_n]$.

This structure is a natural generalisation of the Poisson structure (which corresponds to $n-m=2$) that was introduced by Nambu in 1973 [20] and was extensively studied by Tachtadjan [39].

The most natural example of the Nambu-Poisson structure is the so-called "canonical" Nambu-Poisson structure on $\mathbb{C}^m$ with coordinates $x_1,\ldots,x_m$:

$$\{f_1,\ldots,f_m\} = \mathrm{Jac}(\mathrm{f}_1,\ldots,\mathrm{f}_\mathrm{m}) = \det\frac{\partial(\mathrm{f}_1,\ldots,\mathrm{f}_\mathrm{m})}{\partial(\mathrm{x}_1,\ldots,\mathrm{x}_\mathrm{m})}.$$

We should also remark that formula (3) for the Poisson brackets takes place in more general setting when the polynomials $\mathscr{Q}_i$ are replaced, say, by rational functions, but the resulting brackets are still polynomials. More generally, this formula is valid for the power series rings.

The polynomials $\mathscr{Q}_i$, $i = 1,\ldots,n-2$, are Casimir functions for the bracket (3) and any Poisson structure in $\mathbb{C}^n$ with $n-2$ generic Casimirs $\mathscr{Q}_i$ are written in this form.

The case $n = 4$ in (3) corresponds to the classical (generalised) Sklyanin quadratic Poisson algebra. The Sklyanin algebra is associated with the following two quadrics in $\mathbb{C}^4$:

$$\mathscr{Q}_1 = x_1^2 + x_2^2 + x_3^2, \tag{5a}$$

$$\mathscr{Q}_2 = x_4^2 + J_1 x_1^2 + J_2 x_2^2 + J_3 x_3^2. \tag{5b}$$

The Poisson brackets (3), with $\lambda = 1$ between the affine coordinates, read as follows

$$\{x_i, x_j\} = (-1)^{i+j}\det\left(\frac{\partial\mathscr{Q}_k}{\partial x_l}\right), l \neq i,j,\ i > j. \tag{6}$$

The expression (3) has an advantage before (6), because it is compatible with the more general situations when the intersected varieties are embedded in the weighted projective spaces or in the product of the projective spaces. We will consider an example of such situation below.

The natural question arises as to whether we can extend the brackets (3) or (6) from $\mathbb{C}^n$ to the projective space $\mathbb{C}P^n$.

In fact, we can state the following.

Proposition 2.1. *([23]) Let $X_1, \ldots, X_n$ be coordinates on $\mathbb{C}^n$ defined as an affine part of the corresponding projective space $\mathbb{C}P^n$ with homogeneous coordinates $(x_0 : x_1 : \cdots : x_n)$, $X_i = \frac{x_i}{x_0}$. If the bracket $\{X_i, X_j\}$ can be extended to a holomorphic Poisson structure on $\mathbb{C}P^n$, then the maximum degree of the structure (namely, the maximum degree of monomials in X_i) is 3, and the following identity*

$$X_k\{X_i, X_j\}_3 + X_i\{X_j, X_k\}_3 + X_j\{X_k, X_i\}_3 = 0, \quad i \neq j \neq k, \tag{7}$$

must hold, i.e. $\{X_i, X_j\}_3 = X_i Y_j - X_j Y_i$, with $degY_i = 2$.

More general and precise relations between *quadratic* Poisson structures on affine and projective varieties can be found in [29,31] and in a recent paper [15].

2.2 Sklyanin Elliptic and Poisson Algebras

Here, we remind the notion of Sklyanin elliptic algebras. We shall follow the survey [25] in our notations and we also refer to it as the main source of the results and their proofs of this section. Basically, we reproduce our adaptation [22] of [25].

The elliptic algebras are associative quadratic algebras $Q_{n,k}(Y, \eta)$ which were introduced in the papers [9,21]. Here, Y is an elliptic curve and n, k are integer numbers without common divisors, such that $1 \leq k < n$, while η is a complex number and $Q_{n,k}(Y, 0) = \mathbb{C}[x_1, \ldots, x_n]$.

Let $Y = \mathbb{C}/\Gamma$ be an elliptic curve defined by a lattice $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau \in \mathbb{C}$, $\Im\tau > 0$. The algebra $Q_{n,k}(Y, \eta)$ has generators $x_i, i \in \mathbb{Z}/n\mathbb{Z}$ subject to the relations

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r} x_{i+r} = 0, \tag{8}$$

and has the following properties:

(1) $Q_{n,k}(Y, \eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus \ldots$ such that $Q_\alpha * Q_\beta = Q_{\alpha+\beta}$; here $*$ denotes the algebra multiplication. In other words, the algebras $Q_{n,k}(Y, \eta)$ are $\mathbb{Z}$-graded;

(2) The Hilbert function of $Q_{n,k}(Y, \eta)$ is $\sum_{\alpha \geq 0} \dim Q_\alpha t^\alpha = \frac{1}{(1-t)^n}$.

These properties mean that the algebras $Q_{n,k}(Y, \eta)$ are “PBW-algebras” i.e. they satisfy the Poincaré-Birkhoff-Witt property: $\dim Q_\alpha = \frac{n(n+1)\ldots(n-\alpha+1)}{\alpha!}$.

We consider here the theta-functions $\theta_i(z), i = 1, \ldots, n$, as a base in the space of theta-functions $\Theta_n(\Gamma)$ of order n that are subordinated to the following relations of quasi-periodicity

$$\theta_i(z+1) = \theta_i(z),\ \theta_i(z+\tau) = (-1)^n \exp(-2\pi\sqrt{(}-1)nz)\theta_i(z), i = 0, \ldots, n-1.$$

The theta-function of order 1, $\theta(z) \in \Theta_1(\Gamma)$, satisfies the conditions $\theta(0) = 0$ and $\theta(-z) = \theta(z+\tau) = -\exp(-2\pi\sqrt{(}-1)z)\theta(z)$.

For fixed Y, we see that the algebra $Q_{n,k}(Y,\eta)$ is a flat deformation of the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$. The linear –in η– term of this deformation gives rise to a quadratic Poisson algebra $q_{n,k}(Y)$.

In other words, in the limit as $\eta \to 0$, we obtain a Poisson polynomial algebra $\mathbb{C}[x_1,\ldots,x_n]$ and the *semi-classical limit* of $Q_{n,k}(Y,\eta)$ –denoted by $q_{n,k}(Y)$– is this polynomial algebra equipped with the Poisson bracket

$$\{x_i, x_j\} := \lim_{\eta\to 0} \frac{[x_i, x_j]}{\eta}.$$

For $i \neq j$, it follows from the relations (8) that

$$\{x_i,x_j\} = \left(\frac{\theta'_{j-i}(0)}{\theta_{j-i}(0)} + \frac{\theta'_{k(j-i)}(0)}{\theta_{k(j-i)}(0)} - 2\pi i n\right) x_i x_j + \sum_{r\neq 0, j-i} \frac{\theta_{j-i+r(k-1)}(0)\theta'(0)}{\theta_{kr}(0)\theta_{j-i-r}(0)} x_{j-r}x_{i+r} \tag{9}$$

(see also [15]).

The geometric meaning of the algebras $Q_{n,k}(Y,\eta)$ was underscored in [10,29], where it was shown that the quadratic Poisson structure in the algebras $q_{n,k}(Y)$ associated with the above-mentioned deformation is nothing but the Poisson structure on $\mathbb{P}^{n-1} = \mathbb{P}Ext^1(E,\mathscr{O})$, where E is a stable vector bundle of rank k and degree n on the elliptic curve Y.

In what follows, we denote the algebras $Q_{n,1}(Y,\eta)$ by $Q_n(Y,\eta)$ and the $q_{n,1}(Y)$ by $q_n(Y)$.

2.3 Algebra $Q_n(Y,\eta)$

Construction

For any $n \in \mathbb{N}$, any elliptic curve $Y = \mathbb{C}/\Gamma$ and any point $\eta \in Y$, we construct a graded associative algebra $Q_n(Y,\eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus \ldots$, where $Q_1 = \Theta_n(\Gamma)$ and $Q_\alpha = S^\alpha\Theta_n(\Gamma)$. By construction, $\dim Q_\alpha = \frac{n(n+1)\ldots(n+\alpha-1)}{\alpha!}$. It is clear that the space Q_α can be realised as the space of holomorphic symmetric functions of α variables $\{f(z_1,\ldots,z_\alpha)\}$ such that

$$\begin{aligned} f(z_1+1, z_2,\ldots,z_\alpha) &= f(z_1,\ldots,z_\alpha),\\ f(z_1+\tau, z_2,\ldots,z_\alpha) &= (-1)^n e^{-2\pi i n z_1} f(z_1,\ldots,z_\alpha). \end{aligned} \tag{10}$$

For $f \in Q_\alpha$ and $g \in Q_\beta$, we define the symmetric function $f * g$ of $\alpha+\beta$ variables by the formula

$$\begin{aligned} f * g(z_1,\ldots,z_{\alpha+\beta}) = &\\ \frac{1}{\alpha!\beta!} \sum_{\sigma\in S_{\alpha+\beta}} & f(z_{\sigma_1}+\beta\eta,\ldots,z_{\sigma_\alpha}+\beta\eta) g(z_{\sigma_{\alpha+1}}-\alpha\eta,\ldots,z_{\sigma_{\alpha+\beta}}-\alpha\eta)\times \\ \times & \prod_{\substack{1\le i\le\alpha\\ \alpha+1\le j\le\alpha+\beta}} \frac{\theta(z_{\sigma_i}-z_{\sigma_j}-n\eta)}{\theta(z_{\sigma_i}-z_{\sigma_j})}. \end{aligned} \tag{11}$$

In particular, for $f, g \in Q_1$, we have

$$f * g(z_1, z_2) = f(z_1 + \eta)g(z_2 - \eta)\frac{\theta(z_1 - z_2 - n\eta)}{\theta(z_1 - z_2)} + \\ f(z_2 + \eta)g(z_1 - \eta)\frac{\theta(z_2 - z_1 - n\eta)}{\theta(z_2 - z_1)}.$$

Here, $\theta(z)$ is a theta function of order one.

Proposition 2.2. *If $f \in Q_\alpha$ and $g \in Q_\beta$, then $f * g \in F_{\alpha+\beta}$. The operation $*$ defines an associative multiplication on the space $\oplus_{\alpha \geq 0} Q_\alpha$.*

Main Properties of the Algebra $Q_n(Y, \eta)$

By construction, the dimensions of the graded components of the algebra $Q_n(Y, \eta)$ coincide with those for the polynomial ring in n variables. For $\eta = 0$, the formula for $f * g$ becomes

$$f * g(z_1, \ldots, z_{\alpha+1}) = \frac{1}{\alpha!\beta!} \sum_{\sigma \in S_{\alpha+\beta}} f(z_{\sigma_1}, \ldots, z_{\sigma_\alpha}) g(z_{\sigma_{\alpha+1}}, \ldots, z_{\sigma_{\alpha+\beta}}).$$

This is the formula for the ordinary product in the algebra $S^*\Theta_n(\Gamma)$, that is, in the polynomial ring in n variables. Therefore, for a fixed elliptic curve Y (namely, for a fixed modular parameter τ), the family of algebras $Q_n(Y, \eta)$ is a deformation of the polynomial ring. In particular, there is a Poisson algebra, which we denote by $q_n(Y)$. One can readily obtain the formula for the Poisson bracket on the polynomial ring from the formula for $f * g$, by expanding the difference $f * g - g * f$ in Taylor series with respect to η. It follows from the semi-continuity arguments that the algebra $Q_n(Y, \eta)$ with generic η is determined by n generators and $\frac{n(n-1)}{2}$ quadratic relations. One can prove (see §2.6 in [25]) that this is the case, if η is not a point of finite order on Y, namely $N\eta \notin \Gamma$ for any $N \in \mathbb{N}$.

The space $\Theta_n(\Gamma)$ of the generators of the algebra $Q_n(Y, \eta)$ is endowed with an action of a finite group $\widetilde{\Gamma_n}$ which is a central extension of the group $\Gamma/n\Gamma$ of points of order n on the curve Y. It immediately follows from the formula for the product $*$ that the corresponding transformations of the space $Q_\alpha = S^\alpha\Theta_n(\Gamma)$ are automorphisms of the algebra $Q_n(Y, \eta)$.

Bosonisation of the Algebra $Q_n(Y, \eta)$

The main approach, in order to obtain representations of the algebra $Q_n(Y, \eta)$, is to construct homomorphisms from this algebra to other algebras with simple structure (close to Weil algebras) that have a natural set of representations. Feigin and Odesskii called these homomorphisms *bosonisations*, by analogy with the known constructions of quantum field theory.

Let $B_{p,n}(\eta)$ be a $\mathbb{Z}^p$-graded algebra whose space of degree $(\alpha_1, \ldots, \alpha_p)$ is of the form $\{f(u_1, \ldots, u_p)e_1^{\alpha_1} \ldots e_p^{\alpha_p}\}$; here f ranges over the meromorphic functions of p variables, and $e_1, \ldots, e_p$ are elements of the algebra $B_{p,n}(\eta)$. Moreover, let $B_{p,n}(\eta)$ be generated by the space of meromorphic functions $f(u_1,\ldots,u_p)$ and by the elements $e_1,\ldots,e_p$ with the defining relations

$$\begin{gathered} e_\alpha f(u_1,\ldots,u_p) = f(u_1-2\eta,\ldots,u_\alpha+(n-2)\eta,\ldots,u_p-2\eta)e_\alpha, \\ e_\alpha e_\beta = e_\beta e_\alpha, \quad f(u_1,\ldots,u_p)g(u_1,\ldots,u_p) = g(u_1,\ldots,u_p)f(u_1,\ldots,u_p). \end{gathered} \tag{12}$$

We note that the subalgebra of $B_{p,n}(\eta)$, consisting of the elements of degree $(0,\ldots,0)$, is the commutative algebra of all meromorphic functions of p variables with the ordinary multiplication.

Proposition 2.3. *Let $\eta \in Y$ be a point of infinite order. For any $p \in \mathbb{N}$, there is a homomorphism $\phi_p\colon Q_n(Y,\eta) \to B_{p,n}(\eta)$ that acts on the generators of the algebra $Q_n(Y,\eta)$ by the formula:*

$$\phi_p(f) = \sum_{1\le\alpha\le p} \frac{f(u_\alpha)}{\theta(u_\alpha-u_1)\ldots\theta(u_\alpha-u_p)} e_\alpha. \tag{13}$$

Here, $f \in \Theta_n(\Gamma)$ is a generator of $Q_n(Y,\eta)$ and the product in the denominator is of the form $\prod_{i\ne\alpha}\theta(u_\alpha-u_i)$.

Symplectic Leaves

We recall that $Q_n(Y,0)$ is the polynomial ring $S^*\Theta_n(\Gamma)$. For a fixed elliptic curve $Y=\mathbb{C}/\Gamma$, we obtain the family of algebras $Q_n(Y,\eta)$ which is a flat deformation of the polynomial ring. We denote the corresponding Poisson algebra by $q_n(Y)$. We obtain a family of Poisson algebras, depending on Y, that is, on the modular parameter τ. Let us study the symplectic leaves of this algebra. To this end, we note that, when passing to the limit as $\eta \to 0$, the homomorphism ϕ_p of associative algebras gives a homomorphism of Poisson algebras. Namely, let us denote by $b_{p,n}$ the Poisson algebra formed by the elements $\sum_{\alpha_1,\ldots,\alpha_p\ge 0} f_{\alpha_1,\ldots,\alpha_p}(u_1,\ldots,u_p)e_1^{\alpha_1}\ldots e_p^{\alpha_p}$, where $f_{\alpha_1,\ldots,\alpha_p}$ are meromorphic functions and the Poisson bracket is

$$\{u_\alpha,u_\beta\} = \{e_\alpha,e_\beta\} = 0; \quad \{e_\alpha,u_\beta\} = -2e_\alpha; \quad \{e_\alpha,u_\alpha\} = (n-2)e_\alpha,$$

where $\alpha \ne \beta$.

The following assertion results from Proposition 2.3 in the limit $\eta \to 0$.

Proposition 2.4. *There is a Poisson algebra homomorphism $\psi_p\colon q_n(Y) \to b_{p,n}$ given by the following formula: if $f \in \Theta_n(\Gamma)$, then*

$$\psi_p(f) = \sum_{1\le\alpha\le p} \frac{f(u_\alpha)}{\theta(u_\alpha-u_1)\ldots\theta(u_\alpha-u_p)} e_\alpha.$$

Let $\{\theta_i(u); i \in \mathbb{Z}/n\mathbb{Z}\}$ be a basis of the space $\Theta_n(\Gamma)$, and let $\{x_i; i \in \mathbb{Z}/n\mathbb{Z}\}$ be the corresponding basis in the space of elements of degree one in the algebra $Q_n(Y,\eta)$ (this space is isomorphic to $\Theta_n(\Gamma)$).

For an elliptic curve $Y \subset \mathbb{P}^{n-1}$ embedded by means of theta functions of order n (this is the set of points with the coordinates $(\theta_0(z) : \ldots : \theta_{n-1}(z)))$ we denote by $\mathrm{Sec_p(Y)}$ the variety of p-chords, or $p-$*Secant variety of* Y. By definition, that is the union of projective spaces of dimension $p-1$ passing through p points of Y. It is clear that $\mathrm{Sec_1(Y) = Y}$. Let $C(\mathrm{Sec_p(Y)})$ be the corresponding homogeneous manifold in $\mathbb{C}^n$. Then, $C(\mathrm{Sec_pY})$ consists of the points with coordinates

$$x_i = \sum_{1 \le \alpha \le p} \frac{\theta_i(u_\alpha)}{\theta(u_\alpha - u_1) \ldots \theta(u_\alpha - u_p)} e_\alpha,$$

where $u_\alpha, e_\alpha \in \mathbb{C}$.

Let $2p < n$. Then, one can show that $\dim C(\mathrm{Sec_p(Y)}) = \mathrm{2p}$ and $C(\mathrm{Sec_{p-1}(Y)})$ is the degeneration locus - the variety of singularities of $C(\mathrm{Sec_p(Y)})$. It follows from Proposition 2.4 and from the fact that the Poisson bracket is non-degenerate on $b_{p,n}$ for $2p < n$ and $e_\alpha \neq 0$ that the non-singular part of the variety $C(\mathrm{Sec_p(Y)})$ is a $2p$- dimensional symplectic leaf of the Poisson algebra $q_n(Y)$.

Let n be odd. One can show that the equation defining the manifold $C(\mathrm{Sec_{\frac{n-1}{2}}(Y)})$ is of the form $K_n = 0$, where K_n is a homogeneous polynomial of degree n in the variables x_i. This polynomial is a central function of the algebra $q_n(Y)$.

Let n be even. The manifold $C(\mathrm{Sec_{\frac{n-2}{2}}(Y)})$ is defined by equations $K_n^{(1)} = 0$ and $K_n^{(2)} = 0$, where $\deg K_n^{(1)} = \deg K_n^{(2)} = n/2$. The polynomials $K_n^{(1)}$ and $K_n^{(2)}$ are central in the algebra $q_n(Y)$.

Polishchuk's Description

Polishchuk[1] in his seminal paper [30] defined (as a partial case of his *stable triples* moduli spaces) the space $\mathscr{M}od_{n,k}(Y) = \mathbb{P}(\mathrm{Ext}^1(E, \mathscr{O}_Y)$, where E is a stable vector bundle of degree n and rank k on Y. So, this projective space is associated with the Poisson structure $q_{n,k}(Y)$.

Polishchuk discovered the existence of a birational equivalence

$$\mathscr{M}od_{n,k}(Y) \simeq \mathscr{M}od_{n,-\frac{1}{k+1}}(Y),$$

where an integer $d \equiv -\frac{1}{k+1} \mod \quad n$. The condition $\gcd(n;k) = \gcd(n;k+1) = 1$ provides for the case $n = 5$ that there exists a birational equivalence

$$\mathscr{M}od_{5,1}(Y) \simeq \mathscr{M}od_{5,2}(Y).$$

Let us consider a stable bundle $E \to Y$ of rank $k = 1$ and degree $n = 5$ together with the moduli stack of the triple $(\Phi : \mathscr{O}_Y \to F)$, where $F \to Y$ is a bundle of rank $k+1 = 2$ and degree $n = 5$ with the same determinant as E, i.e. $\det F = \det E$. For an appropriate choice of stability condition, the moduli space $\mathscr{M}od_{5,1}(Y) = \mathbb{P}(\mathrm{Ext}^1(E, \mathscr{O}_Y)$, while some other choice

[1] I am grateful to Brent Pym for enlightened discussion of this point.

of the condition gives the moduli space $\mathbb{P}(H^0(Y,F))$. The birational equivalence $\mathscr{M}od_{5,1}(Y) \dashleftarrow\dashrightarrow \mathbb{P}(H^0(Y,F))$ is given by the wall-crossing of these two stability conditions. Polishchuk claims that $\mathbb{P}(H^0(Y,F))$ is in fact isomorphic to $\mathscr{M}od_{5,-\frac{1}{k+1}}(Y)$. His arguments are based on the Fourier-Mukai transform. It gives us the isomorphism $\mathbb{P}(H^0(Y,F)) \simeq \mathbb{P}(\mathrm{Ext}^1(E',\mathscr{O}_Y)$ with E' to be a stable rank $r' = -\frac{1}{2} \mathrm{mod} \quad 5 = 2$. So, we have the desired transformation $q_{5,1}(Y) \dashleftarrow\dashrightarrow q_{5,2}(Y)$.

3 $q_{5,k}(Y)-$Poisson Tensors

3.1 Why $n = 5$?

In this section, we concentrate our attention on the five-dimensional case. There many different reasons to be interested in this particular case:

- It is the first case when there are two different non-isomorphic families of quantum Feigin-Odesskii elliptic algebras $Q_5(Y,\eta)$ and $Q_{5,2}(Y,\eta')$. It is interesting to compare the corresponding Poisson tensors generated by the Odesskii-Feigin construction;
- The underlying quintic elliptic curves are only *local complete intersections*;
- In the perspective of Integrable systems this comparison is highly desirable. The bihamiltonian properties of $q_{5,2}(Y)$ are still obscure, while the algebra $q_{5,1}(Y)$ is in fact *tri-hamiltonian*, as it was shown by Odesskii in [26];
- Moreover, we still do not know how to relate an integrable system to the algebra $q_{5,2}(Y)$ (as well as to any other odd-dimensional algebras for $n > 3$ and $k \neq 1$).
- The algebras with $n = 5$ generators (together with some known examples in the case $n = 7$) give a good starting point to study the general algebras with *prime* number of generators, p, associated with normal elliptic curves given by Pfaffians.

3.2 H-Invariancy

Let V be a complex vector space of dimension n and $e_0, \ldots, e_{n-1}$ a basis of V. Consider $\varepsilon = e^{\frac{2\pi i}{n}}$ and σ, τ of $GL(V)$ defined by:

$$\sigma(e_m) = e_{m+1};$$

$$\tau(e_m) = \varepsilon^m e_m.$$

As it was recalled in the Introduction, the subspace $H_n \subset GL(V)$ generated by σ and τ is called the Heisenberg of dimension n.

It was observed that the action the maps $x_i \mapsto x_{i+1}$ et $x_i \mapsto \varepsilon^i x_i$, where $\varepsilon^n = 1$, define automorphisms of the algebra $Q_{n,k}(\mathscr{E},\eta)$.

Proposition 3.1. *([27]) Odesskii-Feigin-Sklyanin Poisson algebras satisfy the H-invariance condition.*

We stress that the H-invariant quadratic Poisson structures coincide with Feigin-Odesskii-Sklyanin Poisson algebras in dimensions 3 and 4. In this subsection, we show that these two sets of structures "almost" coincide in dimension 5, and, therefore, the H-invariant Poisson structure is a generalisation of the Odesskii-Feigin Poisson algebras.
The generic form of an antisymmetric H-invariant quadratic matrix is:

$$\begin{pmatrix} 0 & P_1^0 & P_2^0 & -P_2^3 & -P_1^4 \\ -P_1^0 & 0 & P_1^1 & P_2^1 & -P_2^4 \\ -P_2^0 & -P_1^1 & 0 & P_1^2 & P_2^2 \\ P_2^3 & -P_2^1 & -P_1^2 & 0 & P_1^3 \\ P_1^4 & P_2^4 & -P_2^2 & -P_1^3 & 0 \end{pmatrix}, \tag{14}$$

where

$$P_1^k = A_1 x_k x_{1+k} + A_2 x_{2+k} x_{4+k} + A_3 x_{3+k}^2,$$

and

$$P_2^k = B_1 x_k x_{2+k} + B_2 x_{3+k} x_{4+k} + B_3 x_{1+k}^2.$$

The constants A_i and B_i should be determined by the Jacobi identities. For a 5D tensor we have in general 10 independent equations from Jacobi, however the H-invariance reduce only to 2 independent, i.e. $\sum_{l\in\mathbb{Z}_5} P_{0l}\partial_l P_{12} + cyc(i,j,k) = 0$ and $\sum_{l\in\mathbb{Z}_5} P_{0l}\partial_l P_{13} + cyc(i,j,k) = 0$. These PDEs can be reduced to conditions on the coefficients of cubic polynomial in x_i which are equivalent to the system[2]

$$\begin{aligned} {B_2}^2 + 3\,A_1 A_3 + B_1 A_3 + A_2 B_3 &= 0, \\ 2\,{A_3}^2 - 2\,A_2 B_1 - A_1 A_2 + B_2 B_3 &= 0, \\ -{A_2}^2 - 3\,B_1 B_3 + A_1 B_3 + B_2 A_3 &= 0, \\ -2\,{B_3}^2 - 2\,B_2 A_1 + B_1 B_2 - A_2 A_3 &= 0. \end{aligned} \tag{15}$$

Proposition 3.2. *The system (15) admits* 14 *classes of solutions.*

The proof of this proposition was obtained with a help of G.Ortenzi *via* direct computation using the algebraic manipulator Maple. One can write down explicitly all the classes.

We analyse some interesting cases. First of all, the Poisson algebras $q_{5,1}(Y)$ and $q_{5,2}(Y)$ are solutions of system (15) with coefficients

- $$A_1 = -\frac{3}{5a^2} + \frac{a^3}{5}, \quad A_2 = -2a, \quad A_3 = a^2,$$
  $$B_1 = -\frac{1}{5a^2} - \frac{3a^3}{5}, \quad B_2 = 2, \quad B_3 = a^{-1},$$

  for $q_{5,1}(Y)$, and

[2] During a discussion with Sasha Odesskii, we found out that he independently obtained the same result, but it is still unpublished.

- $$A_1 = \frac{2b^2}{5} + \frac{1}{5b^3}, \quad A_2 = b, \quad A_3 = -b^{-1},$$
$$B_1 = -\frac{b^2}{5} + \frac{2}{5b^3}, \quad B_2 = -b^{-2}, \quad B_3 = 1,$$

  for $q_{5,2}(Y)$.

  The "generic" parameters $a, b \in \mathbb{P}^1 \setminus \{0, \infty, (\varepsilon^i(1 \pm \sqrt{5}) : 2), i \in \mathbb{Z}_5, \quad \varepsilon = e^{\frac{2\pi\sqrt{-1}}{5}}\}$. In what follows, we study carefully this couple of Poisson tensors, and we show that they are related by a birational morphism.

There is (apart of this two strata) a cluster of solutions of skew-polynomial type that one can obtain via direct computation using the algebraic manipulator Maple. These solutions correspond to the rational degenerations of elliptic algebras studied by A. Odesskii.

Let us discuss some other interesting cases of solutions.

- The "linear" solution:
$$A_1 = -a^{-1} + 1, \quad A_2 = a^{-1}, \quad A_3 = -a^{-1} - 1,$$
$$B_1 = a^{-1} + 1, \quad B_2 = -2, \quad B_3 = -a^{-1} + 1.$$

  This Poisson structure does not generate any integrable system by the Lenard-Magri scheme, since the Casimir of the structure cannot be represented in a "bihamiltonian" form (i.e. be split as a linear combination of two independent Casimirs).
- We can consider the parameters $a, b \in \mathbb{C}^*$ as a point in $\mathbb{P}^1$. Let, say, a in homogeneous coordinates be a couple $a = (\lambda : \mu)$. The point a parametrises the base elliptic curve (in fact a family of) considering as a schematic intersection of five Klein quadrics in $\mathbb{P}^4$ [16]. Take the cases when $\lambda = 1, \mu = 0$ and $\lambda = 0, \mu = 1$. The corresponding Poisson brackets are examples of quadratic skew polynomial brackets and are associated with two pentagonal configurations in $\mathbb{P}^4$ (degenerations of the schematic intersections of Klein quadrics).
- There are also 10 similar "cuspidal" points in $\mathbb{P}^1$ (they are solutions of the equation $a^{10} - 11a^5 - 1 = 0$).

Explicit form of $q_{5,k}(Y)$

For the tensor $q_{5,1}$, the corresponding Poisson bracket is

$$\{x_i, x_{i+1}\}_{5,1} = \left(-\frac{3}{5a^2} + \frac{a^3}{5}\right) x_i x_{i+1} - 2a\, x_{i+4} x_{i+2} + a^2 x_{i+3}^2,$$
$$\{x_i, x_{i+2}\}_{5,1} = \left(-\frac{1}{5a^2} - \frac{3a^3}{5}\right) x_{i+2} x_i + 2\, x_{i+3} x_{i+4} + \frac{x_{i+1}^2}{a},$$

where $i \in \mathbb{Z}_5$.

For the tensor $q_{5,2}(Y)$, the bracket is

$$\{y_i, y_{i+1}\}_{5,2} = \left(\frac{2}{5}\,b^2 + \frac{1}{5b^3}\right) y_i y_{i+1} + b y_{i+4} y_{i+2} - \frac{{y_{i+3}}^2}{b},$$
$$\{y_i, y_{i+2}\}_{5,2} = \left(-\frac{1}{5}\,b^2 + \frac{2}{5b^3}\right) y_{i+2} y_i - \frac{y_{i+3} y_{i+4}}{b^2} + {y_{i+1}}^2,$$

where $i \in \mathbb{Z}_5$.

In 5 dimensions, the quadratic Poisson tensor admits always a Casimir. The H-invariance allows us to write them explicitly

$$\begin{aligned}
K_5 = {} & c_5 \left({x_0}^5 + {x_1}^5 + {x_2}^5 + {x_3}^5 + {x_4}^5\right) \\
& + c_4 \left({x_0}^3 x_1 x_3 + {x_1}^3 x_0 x_2 + {x_2}^3 x_1 x_4 + {x_3}^3 x_2 x_4 + {x_4}^3 x_0 x_3\right) \\
& + c_3 \left({x_0}^3 x_2 x_4 + {x_1}^3 x_3 x_4 + {x_2}^3 x_0 x_4 + {x_3}^3 x_1 x_0 + {x_4}^3 x_2 x_1\right) \\
& + c_2 \left(x_0 {x_1}^2 {x_4}^2 + x_1 {x_3}^2 {x_0}^2 + x_2 {x_3}^2 {x_1}^2 + x_3 {x_2}^2 {x_4}^2 + x_4 {x_0}^2 {x_3}^2\right) \\
& + c_1 \left(x_0 {x_2}^2 {x_4}^2 + x_1 {x_3}^2 {x_4}^2 + x_2 {x_4}^2 {x_0}^2 + x_3 {x_1}^2 {x_0}^2 + x_4 {x_2}^2 {x_1}^2\right) \\
& + c_0 x_0 x_1 x_2 x_3 x_4,
\end{aligned}$$

where

$$c_5 = -\frac{1}{5} A_3 B_3, \quad c_4 = A_1 A_3, \quad c_3 = -B_1 B_3, \quad c_2 = \frac{1}{2} A_1 A_2 - \frac{1}{2} B_2 B_3,$$
$$c_1 = \frac{1}{2} A_2 A_3 - \frac{1}{2} B_1 B_2, \quad c_0 = A_1^2 - B_1^2 - A_1 B_1 - A_2 B_2,$$

and the constants A_i and B_i satisfy (15).

For example, in the case of $q_{5,1}(Y)$,

$$\begin{aligned}
K_{5,1} = {} & \left(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5\right) \\
& + \left(3 + \frac{1}{a^4}\right) \left(x_0^3 x_1 x_4 + x_1^3 x_0 x_2 + x_2^3 x_1 x_3 + x_3^3 x_2 x_4 + x_2^3 x_0 x_3\right) \\
& - \left(a^4 + \frac{3}{a^2}\right) \left(x_0^3 x_2 x_3 + x_1^3 x_3 x_4 + x_2^3 x_0 x_4 + x_3^3 x_1 x_0 + x_4^3 x_1 x_2\right) \\
& + \left(2a^2 + \frac{1}{a^3}\right) \left(x_0 x_1^2 x_4^2 + x_1 x_2^2 x_0^2 + x_2 x_0^2 x_4^2 + x_3 x_1^2 x_0^2 + x_4 x_1^2 x_2^2\right) \\
& - \left(a^3 - \frac{2}{a^2}\right) \left(x_0 x_2^2 x_3^2 + x_1 x_3^2 x_4^2 + x_2 x_0^2 x_4^2 + x_3 x_1^2 x_0^2 + x_4 x_1^2 x_2^2\right) \\
& - \left(a^5 + 16 - \frac{1}{a^5}\right) x_0 x_1 x_2 x_3 x_4.
\end{aligned} \tag{16}$$

The corresponding central element in the case of $q_{5,2}(Y)$ reads as follows:

$$\begin{aligned} K_{5,2} = &-\frac{1}{b}\left(y_0^5+y_1^5+y_2^5+y_3^5+y_4^5\right) \\ &-\left(\frac{1}{b^5}-3\right)\left(y_0x_2^2y_3^2+y_1y_3^2y_4^2+y_2y_0^2y_4^2+y_3y_1^2y_0^2+y_4y_1^2y_2^2\right) \\ &-\left(b^3+\frac{3}{b^2}\right)\left(y_0y_1^2y_4^2+y_1y_2^2y_0^2+y_2y_0^2y_4^2+y_3y_1^2y_0^2+y_4y_1^2y_2^2\right) \\ &+\left(2b+\frac{1}{b^4}\right)\left(y_0^3y_2y_3+y_1^3y_3y_4+y_2^3y_0y_4+y_3^3y_1y_0+y_4^3y_1y_2\right) \\ &-\left(b^2-\frac{2}{b^3}\right)\left(y_0^3y_1y_4+y_1^3y_0y_2+y_2^3y_1y_3+y_3^3y_2y_4+y_2^3y_0y_3\right) \\ &-\left(b^4+\frac{6}{b}-\frac{1}{b^6}\right)y_0y_1y_2y_3y_4. \end{aligned} \tag{17}$$

It is easy to check that, for any $i \in \mathbb{Z}/5\mathbb{Z}$,

$$\{x_{i+1},x_{i+2}\}\{x_{i+3},x_{i+4}\}+\{x_{i+3},x_{i+1}\}\{x_{i+2},x_{i+4}\}+\{x_{i+2},x_{i+3}\}\{x_{i+1},x_{i+4}\} = \frac{1}{5}\frac{\partial K_5}{\partial x_i},$$

(see [23]).

In other words, the H-invariance implies also the general property

$$P_{ij}P_{kl}+P_{ki}P_{jl}+P_{jk}P_{il} = -2\,\frac{B_2A_1+{B_3}^2}{A_2A_3-B_1B_2}\frac{\partial K_5}{\partial x_m}, \tag{18}$$

where (i,j,k,l,m) is a cyclic permutation of $(0,1,2,3,4)$.
We could interpret this relation as a system of 5 quadric for 6 P_{ij} entries for a fixed polynomial K_5. We see, therefore, that, in contrary to what happens in 3 and 4 dimensions, the Casimir does not encode all the information necessary to the reconstruction of the Poisson tensor. This is related to the fact that 5 is the smallest dimension for which the elliptic Poisson tensors are not of Jacobi type (see [23]).

4 Cremona Transformation

4.1 Generalities about Cremona Transformations

Consider $n+1$ homogeneous polynomial functions φ_i in $\mathbb{C}[x_0,\cdots,x_n]$ of the same degree and not identically zero. One can associate the rational map

$$\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n, [x_0:\cdots:x_n] \mapsto [\varphi_0([x_0,\cdots,x_n):\cdots:\varphi_n([x_0,\cdots,x_n)].$$

The family of polynomials φ_i or φ is called a *birational transformation* of $\mathbb{P}^n$ if there exists a rational map $\psi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\psi \circ \varphi$ is the identity map. A birational transformation is also called a Cremona transformation.

We are basically interested in the very special Cremona transformations $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that there exists a smooth irreducible subvariety $X \subset \mathbb{P}^n$, with a blow-up σ of X in $\mathbb{P}^n$, and the composition $\phi \circ \sigma$ is a *morphism*.

It happens that with $\dim X = 1, 2$ and $n = 4$ there are quite a few such Cremona transformations, and all of them are described in two theorems in the paper of Crauder and Katz [7]. We shall put this description in an appropriate form.

Theorem 4.1. • *A Cremona transformation of* $\mathbb{P}^4$ *which becomes a morphism after the blow-up of a smooth curve* Y*, of degree* d *and genus* g*, is given by the complete linear system of* quadrics *containing* Y*, if and only if* $d = 5$ *and* $g = 1$*. In other words* Y *is an* elliptic normal[3] *quintic.*

- *Let* X *be a surface blown up in* $\mathbb{P}^4$*. The Cremona transformation given by the complete linear system of* cubics *containing* X *becomes a morphism after the blow-up of* X*, if and only if* X *is a quintic elliptic* scroll[4] $X = \mathbb{P}_Y(E)$ *with* $e(E) = -1$*.*
- *Here,* $E \to Y$ *is a rank* 2 *vector bundle. If* $X = \mathbb{P}(\mathscr{E})$*, then there is the unique non-trivial extension*

$$0 \to \mathscr{O}_Y(-\eta) \to E \to \mathscr{O}_Y \to 0, \tag{19}$$

where $\eta \in Y$ *is a fixed point on the elliptic curve. The surface* X *is isomorphic to* $S^2(Y)$ *and the map* $p : X \to Y$ *is nothing but the Abel-Jacobi map which has a structure of projective bundle* $\mathbb{P}(E) \to Y$*.*
- *The latter ("cubic") Cremona transformation of* $\mathbb{P}^4$ *is inverse to the "quadric" Cremona transformation in* $\mathbb{P}^4$ *given by the elliptic normal quintic* Y*.*

Remark We do not use the terminology of the XIX-th and early XX-th century geometers worked out the Cremona transformations. Due to the remark in ([17], p.422), a modern "dictionary" is available. For example, the linear system of polynomials $\varphi_0, \ldots, \varphi_n$ on $\mathbb{P}^n$ such that the generic fiber of the resulting rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is discrete usually was called *sub-homoloidal system* (see [32]). Thus, the transformations described in the Crauder-Katz Theorem 4.1 are nothing but the classical "quadro-cubic" Cremona transformations of $\mathbb{P}^4$ discovered in [33].

Now, we study these linear systems of quadrics and cubics, so we are concentrated on the case $n = 4$.

4.2 Case $n = 4$. Cremona transformation and Elliptic Curves.

Let the polynomials $q_0, \ldots, q_4 \in H^0(\mathscr{O}_{\mathbb{P}^4}(2))$ define a rational map Φ

[3] This means that the embedding of Y in $\mathbb{P}^4$ is given by complete linear system.
[4] A ruled degree 5 surface in $\mathbb{P}^4$.

$$\begin{tikzcd}
\widehat{\mathbb{P}^4} \arrow[d, dotted] \arrow[dr] & \\
\Phi : \mathbb{P}^4 \arrow[r] & \mathbb{P}^4
\end{tikzcd}$$

where $\widehat{\mathbb{P}^4}$ is a blow-up of $\mathbb{P}^4$.

We denote by $||M_{ij}||_{0 \le i,j \le 4}$ any matrix of *syzygies*, i.e. any matrix for which

$$\sum_{i=0}^{4} q_i M_{ij} = 0,$$

for any $0 \le j \le 4$.

Following [17], we call any such *dominant* map a *Φ Cremona transformation* of $\mathbb{P}^4$.

Example 4.1. *Let* $q_i = {x_i}^2 + a x_{i+2} x_{i+3} - \frac{x_{i+1} x_{i+4}}{a}, \quad i \in \mathbb{Z}_5, \quad a \in \mathbb{C}^*$. *We consider the following alternating* 5×5 *matrix* $M = ||M_{ij}||$:

$$\begin{pmatrix} 0 & -x_3 & -ax_1 & ax_4 & x_2 \\ x_3 & 0 & -x_4 & -ax_2 & ax_0 \\ ax_1 & x_4 & 0 & -x_0 & -ax_3 \\ -ax_4 & ax_2 & x_0 & 0 & -x_1 \\ -x_2 & -ax_0 & ax_3 & x_1 & 0 \end{pmatrix}. \tag{20}$$

It is easy to check that

$$\sum_{i=0}^{4} q_i M_{ij} = 0, \tag{21}$$

for any $0 \le j \le 4$.

This matrix of syzygies is called *Klein matrix of linear forms.* It is well-known (see [11]) that any Heisenberg invariant elliptic curve of odd degree, containing the unique point p of the form:

$$p = (a_0 \equiv 0 : a_1 : a_2 : \ldots : -a_2 : -a_1),$$

has a Klein matrix of the form $M_{ij} = a_{i-j} x_{i+j}$. In our case (4.1),

$$p = (0 : -1 : -a : a : 1),$$

and it goes back to Klein's works that the polynomials $q_i, \quad 0 \le i \le 4$ –the generators of the homogeneous ideal $\mathscr{I}(q_0, \ldots, q_4) \subset \mathbb{C}[\mathbb{P}^4]$– are 4×4 *Pfaffians* of the Klein matrix (20). The elliptic curve family Y_a is cut off by five quadrics-Pfaffians $q_i, 0 \le i \le 4$:

$$Y_a = \bigcap_{i=0}^{4} q_i.$$

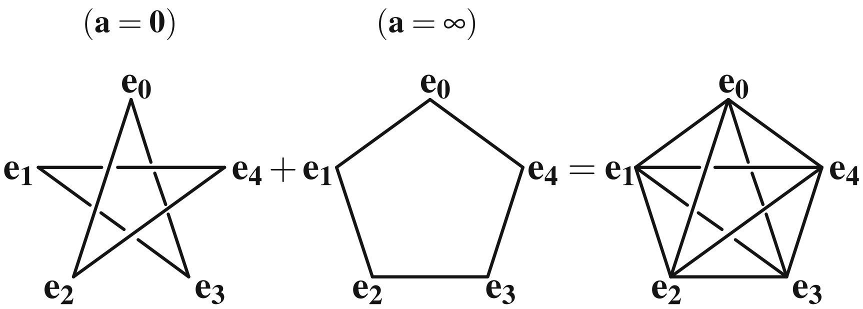
If $\mathscr{J}_{Y_a}$ is the ideal sheaf of a "generic" (=smooth) curve then the following exact sequence takes place

$$0 \to \mathscr{J}_{Y_a} \to \mathscr{O}_{\mathbb{P}^4}(2) \to \mathscr{O}_{Y_a}(2) \to 0.$$

Here, we identify $H^0(\mathscr{O}_{\mathbb{P}^4}(2))$ with $S^2(\Theta_{5,c}(\varGamma))$.

We shall basically consider smooth generic (with respect to the parameter a) genus one curves, unless otherwise indicated. We precise the meaning of the word "generic".

It should be remarked (see [5] and [16]) that the curves Y_a are smooth, if a is non-exceptional in the following sense. In fact, ([5]) the family Y_a extends to $\mathbb{P}^1$-family $Y_{p,q}$ $(a = (p:q))$, which is smooth, if $(p:q) \neq (1:0), (0:1), (\varepsilon^i(1 \pm \sqrt{5}) : 2), i \in \mathbb{Z}_5, \quad \varepsilon = e^{\frac{2\pi\sqrt{-1}}{5}}$. These exceptional points correspond to the vertices of the *Klein icosahedron* inside $\mathbb{S}^2 = \mathbb{P}^1$ and the associated singular curves forms *pentagons* (the following figures appeared for the first time in [5,16]):



We shall finish this description with the well-known fact that the union of the families $Y_{p,q}$ over $\mathbb{P}^1$ is an algebraic surface

$$S_{15} = \bigcup_{(p:q)\in\mathbb{P}^1} Y_{p,q}. \tag{22}$$

This surface is a determinantal surface of degree 15 and it is smooth out of the above singular curve pentagon vertices. This surface (after a normalisation) is isomorphic to the Shioda modular surface $S(5)$, and it is an *elliptic surface* over the rational modular curve $X(5) = \mathbb{P}^1$ whose smooth fibers are smooth families $Y_{p,q}$ (for all these facts see [4,5,16]).

4.3 Quadro-Cubic Cremona Transformations

In this subsection, we show that the two five-dimensional elliptic algebras are related by some Cremona transformation.

The proof of the below statement was obtained with the help of G. Ortenzi using the MAPLE manipulator.

Let us consider the Cremona transformation $\Phi : \mathbb{P}^4(x) \to \mathbb{P}^4(z)$ given by the following homogeneous polynomial functions

$$
\begin{array}{l}
z_0 = q_0 = {x_0}^2 + a x_2 x_3 - \frac{x_1 x_4}{a}, \\
z_1 = q_1 = {x_1}^2 + a x_3 x_4 - \frac{x_2 x_0}{a}, \\
z_2 = q_2 = {x_2}^2 + a x_4 x_0 - \frac{x_3 x_1}{a}, \\
z_3 = q_3 = {x_3}^2 + a x_0 x_1 - \frac{x_4 x_2}{a}, \\
z_4 = q_4 = {x_4}^2 + a x_1 x_2 - \frac{x_0 x_3}{a}.
\end{array}
\tag{23}
$$

In general, this transformation does not map the $H-$invariant Poisson structure of $q_{5,1}(Y)$ to another H-invariant quadratic Poisson structure on $\mathbb{P}^4$.

Proposition 4.1. • *Set*

$$
a = -\frac{1}{\lambda}. \tag{24}
$$

Then, the map (23) maps the Poisson structure $q_{5,1(Y)}$, with parameter λ, to the structure $q_{5,2}(Y)$.

- *The Casimir of $q_{5,1}(Y)$ coincides with the determinant of the Jacobian of (23).*
- *The inverse Cremona transfomation $\Phi^{-1} : \mathbb{P}^4(z) \to \mathbb{P}^4(x)$ is the cubic Poisson morphism between $q_{5,2}(Y)$ and $q_{5,1(Y)}$ which can be explicitly written as:*

$$
\begin{aligned}
x_i = \mathscr{Q}_{i,\lambda}(z) = & \frac{z_{i+2}{z_{i+4}}^2 + {z_{i+1}}^2 z_{i+3}}{\lambda} - \lambda\left(z_{i+1}{z_{i+2}}^2 + {z_{i+3}}^2 z_{i+4}\right) \\
& + {z_i}^3 - z_i\left(\frac{z_{i+1}z_{i+4}}{\lambda^2} + \lambda^2 z_{i+2}z_{i+3}\right),
\end{aligned}
\tag{25}
$$

for $i = 0, 1, 2, 3, 4$.

- *The Casimir of $q_{5,2}(Y)$ coincides with the determinant of some matrix $L(z, \lambda)$ which can be reconstructed from (25) .*

Remark 4.1. *The explicit determinantal formula for the Casimir of $q_{5,2}(Y)$ will be done in the next section.*

The constraint (24) is calculated as follows. We know that the structure of the resulting $H-$invariant quadratic Poisson algebra should have a similar structure as in the case $q_{5,1}(Y)$ but with different coefficients. We impose this constraint and we calculate all the P_{ij} as a function of the parameter λ of the $q_{5,1}(Y)$ algebra and the parameter a of the change of variables. This is possible if and only if a and λ solve the system

$$
\begin{cases}
-a^3\lambda + 4\,\lambda^4 a + 2\,\lambda^5 a^2 + 2\,\lambda^3 - 2\,a^2 + a^6\lambda^4 = 0, \\
-1 + 2\,a^2\lambda^2 - a^3\lambda^3 + 2\,a\lambda = 0.
\end{cases}
$$

The second equation is reducible:

$$
(a\lambda + 1)(a^2\lambda^2 - 3a\lambda + 1) = 0.
$$

If $a = -\frac{1}{\lambda}$, the system is identically solved and we find a class of Poisson tensor depending nonlinearly on a parameter. If $a = \frac{3\pm\sqrt{5}}{2\lambda}$, namely the solutions of the second equation, then using the first equation we fix 5 possible values of λ and we find some non-parametric "isolated" Poisson tensors which correspond to the points on the Klein icosahedron.

4.4 A Remark on the Compatibility

The explicit values of the parameters are

$$a = \frac{3 \pm \sqrt{5}}{2\lambda} \qquad \textit{with} \qquad \lambda^5 = -\frac{11 \pm 5\sqrt{5}}{2}. \tag{26}$$

We check only in one case where these structures admit 5 independent *quasi-Casimirs* [23]. These structures are in fact some linear Poisson pencils related to $q_{5,1}(Y)$: the tensors $q_{5,1(Y)}$ and $q_{5,2}(Y)$ depend in a nonlinear way on the parameter λ, therefore they do not generate a bi-Hamiltonian pencil with respect to all λ.

There is a natural question: is there some value of λ for which the tensors in the family $q_{5,1}(Y)$ are compatible?
The unexpected answer is **yes**.

Proposition 4.2. *Let us denote $q_{5,1}(Y)$ by $q^{\lambda}_{5,1}(Y)$ making explicitly its dependence on the parameter λ. Therefore, $q^s_{5,1} + bq^r_{5,1}$ is a Poisson tensor $\forall b \in \mathbb{C}$ if and only if*

$$\frac{r}{s} = \pm a\lambda,$$

where a and λ satisfies the (26). It can be proved by a direct computation.

Remark 4.2. *In [26], the author points out a different bi-Hamiltonian property of the $q_{n,1}(Y)$ Poisson algebras: Every structure, by means of a non-polynomial change of variables related to the connection between theta- and Weierstrass$\mathscr{P}$-functions, can be written as a combination of three compatible Poisson structures. The role of linear parameters is played by the g_2 and g_3 coefficients of the Weierstrass canonical equation. It seems that our results justify the similar realisation at least for the case of $q_{5,2}(Y)$ elliptic Feigin-Odesskii-Sklyanin algebra.*

5 Klein and Moore Syzygies and Poisson Structures

5.1 Klein Syzygy and Free Resolution Complex.

It is easy to verify that the Klein matrix M associated with Y_a has rank 2 on Y_a and vice versa; namely, if the family Y_a is rank-2 locus of some matrix $\widehat{M}$, then this matrix is the Klein matrix of linear forms of Y_a (Corr. 6.4 in [14]).

From the algebraic point of view, the set Y_5 of genus one curves of degree 5 is an affine five-dimensional space, and its coordinate ring $\mathbb{C}[Y_5]$ is a polynomial ring with 5 variables.

A genus one degree 5 normal curve f is described by a collection of 5 quadratic polynomials $q_i \in Y_5$, $i = 0, \dots 4$, defining an ideal $\mathscr{I}_f \subset \mathbb{C}[x_0, \dots, x_4]$. Using the Kleinian syzygy matrix of *linear* forms $M_{ij} = a_{i-j}x_{i+j}$, one can generate the ideal $\mathscr{I}_f = \mathscr{I}(q_0, \dots, q_4)$, the vector P of 4×4 Pfaffians $\mathscr{Q} = (q_0, \dots, q_4)$ and a a minimal free resolution complex [4,13]:

$$0 \to A(-5) \xrightarrow{\mathscr{Q}^t} A(-3)^5 \xrightarrow{f} A(-2)^5 \xrightarrow{\mathscr{Q}} \mathbb{C}[Y_5] = \mathbb{C}[x_0, \dots, x_4] \to 0, \tag{27}$$

with homology concentrated in the last term: $H_0 = \mathbb{C}[Y_5]/\mathscr{I}_f$. The notation $A(d)$ means a graded $\mathbb{C}[Y_5]$-module with k-th graded component A^{k+d}. Under some generic condition the complex (27) is also a free minimal resolution of $\mathbb{C}[Y_5]/\mathscr{I}_f$. The exactness of (27) in the term $A(-2)^5$ gives an another way to express the Klein syzygy property (21):

$$\mathscr{Q} \cdot f = \sum_{k=0}^{4} q_i m_{ij} = 0.$$

5.2 Poisson Structure on $\mathbb{P}^4$ from Syzygies

Now, we hint how to obtain the elliptic Poisson Feigin-Odesskii-Sklyanin algebras directly from the construction similar to above Klein syzygy description replacing the skew-symmetric matrix of *linear forms* by some skew-symmetric matrix of *quadratic forms*. With this aim, we remind that for any normal genus 1 curve Y, there exists a naturally attached alternating matrix of quadrics $\Omega = ||\Omega_{ij}||$. This matrix appears in the works of T. Fisher (for example, see [12,13]). In particular, he proved ([13], Th. 1.1 (i)) that if there exists a function K such that the secant variety $\mathrm{Sec}(Y)(:= \mathrm{Sec}^2(Y))$ is the level $\{K = 0\}$, then there exists a minimal free resolution of the naturally graded polynomial ring $A = \mathbb{C}[x_0, \ldots, x_4]$ by graded A-*modules* similar to (27):

$$0 \to A(-10) \xrightarrow{\nabla^t} A(-6)^5 \xrightarrow{\Omega} A(-4)^5 \xrightarrow{\nabla} A, \tag{28}$$

where Ω is a 5×5 skew-symmetric matrix of quadratic forms, ∇ is the nabla operator,

$$\nabla(f) := \left(\frac{\partial f}{\partial x_0}, \ldots \frac{\partial f}{\partial x_4}\right),$$

and the notation $A(d)$ means a graded A-module with graded component as it was explained above. The exactness of (28) in the term $A(-4)^5$ gives

$$\nabla(K) \cdot \Omega = \sum_{k=0}^{4} \frac{\partial K}{\partial x_k} \Omega_{kj} = 0,$$

for $\mathrm{Sec}(Y) = \{K = 0\}$ (Lemma 4.5 in [13]). Moreover, one can readily verify (Lemma 4.7 (i) ibid.) that the 4×4 Pfaffians of Ω are (up to a constant $c \neq 0$) the partial derivatives of K:

$$(-1)^k \mathrm{pf}(\Omega^{(k)}) = c \frac{\partial K}{\partial x_k}, \quad k = 0, \ldots, 4.$$

So, one can say that the Ω is the alternating matrix of quadratic forms such that $\nabla(f)\Omega \equiv 0$ on the anticanonical divisor $K(Y)$ for all $f \in \mathscr{I}(Y)$ of rank $\Omega = 2$ on Y.

$$\Omega = \begin{pmatrix} 0 & \alpha_3 & \beta_1 & -\beta_4 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_4 & \beta_2 & -\beta_0 \\ -\beta_1 & -\alpha_4 & 0 & \alpha_0 & \beta_3 \\ \beta_4 & -\beta_2 & -\alpha_0 & 0 & \alpha_1 \\ \alpha_2 & \beta_0 & -\beta_3 & -\alpha_1 & 0 \end{pmatrix}, \tag{29}$$

where

$$\alpha_i = 5\lambda^4\mu x_i^2 - 10\lambda^3\mu^2 x_{i-1}x_{i+1} + \left(\lambda^5 - 3\mu^5\right) x_{i-2}x_{i+2},$$
$$\beta_i = 5\lambda\mu^4 x_i^2 - \left(3\lambda^5 + \mu^5\right) x_{i-1}x_{i+1} + 10\lambda^2\mu^3 x_{i-2}x_{i+2}.$$

Theorem 5.1. *The matrix Ω (after an appropriate rescaling) coincides with the matrix of the Poisson tensor for $q_{5,1}(Y)$.*

Proof. The direct verification and a comparison with (16) in Chap. 7 of [12] provides that the matrix

$$\alpha_i = c\left(\{x_{i+2}, x_{i+3}\}\right) = c\left((\frac{a^3}{5} - \frac{3}{5a^2})x_{i+2}x_{i+3} - 2ax_{i+1}x_{i+4} + a^2x_i^2 \right),$$
$$\beta_i = c\left(\{x_{i+1}, x_{i+4}\}\right) = c\left(-(\frac{1}{5a^2} + \frac{3a^3}{5})x_{i+1}x_{i+4} + 2x_{i+2}x_{i+3} + ax_i^2 \right).$$

In order to get the explicit coincidence of α_i with $\{x_{i+2}, x_{i+3}\}_{5,1}$ and β_i with $\{x_{i+1}, x_{i+4}\}$, we should take $c = 5\lambda^3\mu^2$ and denote $\frac{\lambda}{\mu}$ by a. In other words, the matrix (29) appears as the Poisson tensor matrix for $q_{5,1}(Y)$, where Y is (for generic a) given by the Example 4.1. We shall denote the quintic polynomial by $K := K_{5,1}(x)$. This polynomial is given by the determinant of the Jacobian matrix:

$$K_{5,1}(x) = \det\|\frac{\partial q_i}{\partial x_j}\|.$$

Remark 5.1. *The secant variety* $\mathrm{Sec}(Y) = \{K_{5,1}(x) = \det\|\frac{\partial q_i}{\partial x_j}\| = 0\}$ *is the quintic 3-fold in* $\mathbb{P}^4$ *which describes the "maximal symplectic leaf"* M^2 *mentioned in the Introduction.*

The above construction admits the following inversion or a "duality" in the framework of the Cremona transformations of Subsect. 4.3.

5.3 Moore Syzygies and Corresponding 3-Folds.

We observe that the defining cubic relations (25) can be interpreted as 4×4-minors of some 5×5-matrix $L_\lambda(z)$ where $\lambda \in \mathbb{C}^*$ and the matrix entries

$$||L_{ij}||_{0\leq i,j\leq 4} := ||(z_{2i-j})_\lambda||.$$

(We do not precise the dependence on λ because the matrices $L_\lambda(z)$ are determined up to scalar only.) We shall be using the term *Moore-like syzygy matrix*[5] for the matrices $L_\lambda(z)$.

The cubics (25) define an ideal $\mathscr{I}_{\mathscr{Q}_\lambda} \subset \mathbb{C}[z_0, \ldots, z_4]$. The ideal $\mathscr{I}_{\mathscr{Q}_\lambda}$ defines a ruled surface of degree 5 - the quintic elliptic scroll $\mathscr{Q}_\lambda \subset \mathbb{P}^4(z)$.

[5] In fact the entries of the Moore-like syzygy matrix depends on $\imath(z)$ where $\imath(z_i) = z_i$ is the involution. But we need the matrix $L_\lambda(z)$ to describe the surfaces of symplectic foliations which are $\imath-$invariants (compare with [4]).

In order to visualise it, we remind the following construction: By the Lemma 4.4 in [4], we have the free resolution complex for $\mathbb{C}[\mathscr{Q}_\lambda]/\mathscr{I}_{\mathscr{Q}_\lambda}$ similarly to (27), namely

$$0 \to A(-5) \xrightarrow{\mathscr{P}^t} A(-4)^5 \xrightarrow{(L_\lambda(z))^t} A(-3)^5 \xrightarrow{\mathscr{P}} \mathbb{C}[\mathscr{Q}_\lambda] = \mathbb{C}[z_0, \dots, z_4] \to 0, \quad (30)$$

where $\mathscr{P} := (p_0, \dots, p_4)$.

The exactness in the term $A(-4)^5$ means that $L_\lambda(z))^t \mathscr{P}^t = 0$ or, in matrix form,

$$\begin{pmatrix} 0 & -\lambda z_2 & \lambda^{-1} z_4 & -\lambda^{-1} z_1 & \lambda z_3 \\ \lambda z_4 & 0 & -\lambda z_3 & \lambda^{-1} z_0 & -\lambda^{-1} z_2 \\ -\lambda^{-1} z_3 & \lambda z_0 & 0 & -\lambda z_4 & \lambda^{-1} z_1 \\ \lambda^{-1} z_2 & -\lambda^{-1} z_4 & \lambda z_1 & 0 & -\lambda z_0 \\ -\lambda z_1 & \lambda^{-1} z_3 & -\lambda^{-1} z_0 & \lambda z_1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = 0. \quad (31)$$

By a straightforward calculation, we show that $\det L_\lambda(z)) = 0$. We take the 4×4 minors in (31) (following [4] and [19]):

$$M_{ij}^{(4\times 4)}(L_\lambda(z)) = z_i \mathscr{Q}_{j,\lambda}(z), \quad (32)$$

where cubics $\mathscr{Q}_{j,\lambda}(z)$ are generators of the ideal $\mathscr{I}_{\mathscr{Q}_\lambda}$ and defines the inverse "cubic" Cremona transformation (25) as in the classical Semple's quadro-cubic setting.

But, the left-hand side variables x_j, $j = 0, \dots, 4$ of (25) satisfy the quadric relations

$$x_j{}^2 - a^{-1} x_{j+2} x_{j+3} + a x_{j+1} x_{j+4} = 0, \quad i \in \mathbb{Z}_5, \quad a \in \mathbb{C}^*. \quad (33)$$

Taking in (25) the constraint $\lambda = a^{-1}$, we obtain the system of quadric relations (33) whose entries are the cubic polynomial in z

$$\mathscr{Q}_{j,a^{-1}}(z)^2 - a^{-1} \mathscr{Q}_{j+2,a^{-1}}(z) \mathscr{Q}_{j+3,a^{-1}}(z) + a \mathscr{Q}_{j+1,a^{-1}}(z) \mathscr{Q}_{j+4,a^{-1}}(z) = 0. \quad (34)$$

The equation $\det L_\lambda(z) = 0$ defines a degree 5 polynomial $\tilde{K}(z, a)$ and also a (H_5−Heisenberg and involution invariant) cubic 3-fold in $\mathbb{P}^4$,

$$\tilde{K}(z, a) := \det L_\lambda(z)) = 0,$$

or, explicitly,

$$\begin{aligned} \tilde{K}(z, a) = (a^5 + 6 - a^{-5}) z_0 z_1 z_2 z_3 z_4 + \\ \sum_{j=0}^{4} \left[z_j^5 + (a^4 + 3a^{-1}) z_j z_{j+2}^2 z_{j+3}^2 + (a^{-4} - 3a) z_j z_{j+1}^2 z_{j+4}^2 \right] + \\ + \sum_{j=0}^{4} \left[(a^3 - 2a^{-2}) z_j^3 z_{j+2} z_{j+3} - (2a^2 + a^{-3}) z_j^3 z_{j+1} z_{j+4} \right] = 0. \end{aligned}$$

Comparing with the expression of the Casimir quintic polynomial $K_{5,2}$ for $q_{5,2}(Y)$, we see that (taking $y = z$, $b = a$)

$$\tilde{K}(z,a) = -aK_{5,2}(z)$$

and (for generic $a \in \mathbb{C}^*$) they define the same 3-fold in $\mathbb{P}^4$.

We obtain (in a accordance with the Corollary 4.5 from [4]) that the elliptic quintic scroll, defined by the $q_{5,2}(Y)$ Casimir level $K_{5,2}(z) = 0$, is exactly the $\mathscr{Q}_{a^{-1}}$ given by the system of cubics $\mathscr{Q}_{j,\lambda}(z)$ from (25) with the constraint $a\lambda = 1$.

5.4 Poisson Structure on $\mathbb{P}^4$ from Moore Syzygies

Now, we can proceed with the quintic polynomial $\tilde{K}(z,a)$ (or directly with $K_{5,2}(z)$) in the same fashion as in the case of $K_{5,1}(z)$.

There is an analog of the resolution (28) and a skew-symmetric 5−matrix $\tilde{\Omega}$ such that

$$\nabla(K_{5,2}) \cdot \Omega = \sum_{k=0}^{4} \frac{\partial K}{\partial z_k} \Omega_{kj} = 0,$$

The gradient $\nabla K_{5,2}$ of the polynomial $K_{5,2}$ also admits a "Pfaffian" representation

$$(-1)^k \mathrm{pf}(\tilde{\Omega}^{(k)}) = (\tilde{c}) \frac{\partial K_{5,2}}{\partial z_k}, \quad k = 0, \ldots, 4.$$

We should assume (again, by Fisher's Lemma 4.5 in [13]) that there exist some *other elliptic curve* $\tilde{Y}_a$ for which the secant variety $\mathrm{Sec}(\tilde{Y}_a) = 0$ is given by the quintic polynomial zero level $\{K_{5,2}(a,z) = 0\}$.

The curve $\tilde{Y}_a$ can be described (for generic choice of the parameter a) by 4×4 sub-Pfaffians of some alternating Moore-like syzygy 5×5 matrix L_a, the polynomials p_i generate the coordinate ring ideal $\mathscr{I}_{\tilde{Y}_a}$ for an elliptic quintic curve $\tilde{Y}_a$.

$$\frac{\partial K_{5,2}}{\partial z_i} = p_i(q_0, \ldots q_4), \quad i = 0,1,2,3,4,$$

and $p_0, \ldots, p_4$ are Pfaffians of the syzygy matrix L_a^t for $\tilde{Y}_a$. We shall consider the quadrics $q_0, \ldots, q_4$ as homogeneous coordinates $(x_0 : \ldots : x_4)$ in the projective space $\mathbb{P}^4$.

Then, $p_i = p_i(x_0, \ldots, x_4), \quad i = 0,1,2,3,4$, and the ideal $\mathscr{I}_{\tilde{Y}_a}$ is given by

$$p_i = {x_i}^2 - a^{-1} x_{i+2} x_{i+3} + a x_{i+1} x_{i+4}, \quad i \in \mathbb{Z}_5, \quad a \in \mathbb{C}^*.$$

A straightforward computation verifying by the "chain rule" shows that

$$\sum_{i=0}^{4} \frac{\partial p_i(q_0, \ldots q_4)}{\partial x_i} \frac{\partial q_i(z_0, \ldots, z_4)}{\partial z_j} = 0.$$

We consider the following alternating 5×5 matrix $\tilde{\Omega}$ such that $\operatorname{rank} \tilde{\Omega} = 2$ on the curve $\tilde{Y}_a$:

$$\tilde{\Omega} = \begin{pmatrix} 0 & \tilde{\alpha}_3 & \tilde{\beta}_1 & -\tilde{\beta}_4 & -\tilde{\alpha}_2 \\ -\tilde{\alpha}_3 & 0 & \tilde{\alpha}_4 & \tilde{\beta}_2 & -\tilde{\beta}_0 \\ -\tilde{\beta}_1 & -\tilde{\alpha}_4 & 0 & \tilde{\alpha}_0 & \tilde{\beta}_3 \\ \tilde{\beta}_4 & -\tilde{\beta}_2 & -\tilde{\alpha}_0 & 0 & \tilde{\alpha}_1 \\ \tilde{\alpha}_2 & \tilde{\beta}_0 & -\tilde{\beta}_3 & -\tilde{\alpha}_1 & 0 \end{pmatrix}, \tag{35}$$

with

$$\tilde{\alpha}_i = \tilde{c}\left(\{x_{i+2}, x_{i+3}\}\right) = \tilde{c}\left((\frac{2}{5a^2} + \frac{a^3}{5}) x_{i+2}x_{i+3} + x_{i+1}z_{i+4} - a^{-1}x_i^2 \right),$$

$$\tilde{\beta}_i = \tilde{c}\left(\{x_{i+1}, x_{i+4}\}\right) = \tilde{c}\left(-(\frac{a^2}{5} - \frac{2}{5a^3}) x_{i+1}x_{i+4} - a^{-2}x_{i+2}x_{i+3} + x_i^2 \right).$$

Theorem 5.2. *The matrix (5.2) (after some appropriate rescaling) coincides with the matrix of the Poisson tensor for* $q_{5,2}(\tilde{Y}_a)$.

Here, $a \in \mathbb{C}^*$ and $\tilde{Y}_a$ is given by the set of the quadrics (33):

$$x_i = q_i(z) = z_i^2 - a z_{i+2} z_{i+3} + a^{-1} z_{i+1} z_{i+4}, \qquad i \in \mathbb{Z}_5.$$

The elliptic quintic scroll $\mathscr{Q}_a(\tilde{Y})(x)$ is given by the set of cubics

$$z_i = \mathscr{Q}_a(\tilde{Y})(x) = x_i^3 + a(x_{i+1}^2 x_{i+3} + x_{i+2}x_{i+4}^2) - a^{-1}(x_{i+1}x_{i+2}^2 + x_{i+3}^2 x_{i+4}) -$$
$$-a^2 x_i x_{i+1} x_{i+4} - a^{-2} x_i x_{i+2} x_{i+3}, \qquad i \in \mathbb{Z}_5.$$

Thus, the Poisson properties of the quadro-cubic Cremona transformations in P^4 give two pairs of geometric objects and some duality between them: The first pair is a normal quintic elliptic curve $Y_a \subset \mathbb{P}^4(x)$ parametrised by some point $a \in \mathbb{P}^1$ and some elliptic quintic scroll - a ruled degree 5 surface $\mathscr{Q}(Y_a)$ which is in generic point may be identified with $X = S^2(Y)$.

The "dual pair" of geometric objects contains again an elliptic curve $\tilde{Y}_a \subset \mathbb{P}^4(z)$ of degree 15 and a singular elliptic scroll S of degree 15. By analogy with (22), one can consider a degree 45 surface

$$S_{45} = \bigcup_{(a) \in \mathbb{P}^1} \tilde{Y}_a. \tag{36}$$

The elliptic curves Y_a and $\tilde{Y}_a$ are isomorphic under some birational map $v_+ : S_{15} \dashrightarrow S_{45}$ (the Horrocks-Mumford map).

The geometry of this duality was studied and described in [4] and [3]. We shall use the Proposition 4.12 in [4] to explain it:

Proposition 5.1. • *Let Y_a be a normal elliptic quintic curve with generic value $a \in C^*$. Any elliptic scroll may be identified with a symmetric product of elliptic curves (in our case $\mathscr{Q}_a(Y)$ canonically with $S^2(Y_a)$).*

- *There is an image of the elliptic quintic curve* Y_a *denoted by* $\tilde{Y}_a$ *under the Horrocks-Mumford map*

$$v_+ : \mathbb{P}^4(x) \dashrightarrow \mathbb{P}^4(z)$$

$$z_i = v_+(x) = x_{i+2}x_{i+4}^2 - x_{i+1}^2 x_{i+3}, \quad i \in \mathbb{Z}_5$$

  as a degree 15 curve.
- *In a smooth generic (non-torsion) point* $p \in Y_a$ *the quintic hypersurface defined by the Jacobian matrix quadrics* $\det M_p(z) = 0$ *is the secant variety* $\mathrm{Sec}(\mathrm{Y_a})$*, and the quintic hypersurface defined by the Moore syzygies* $\det L_q(z) = 0$ *is* $\mathrm{Trisec}(S^2(Y_a))$ *– the trisecant variety* $\mathcal{Q}_{a^{-1}}(Y)$*.*
- *Let* q *be a non-torsion order 5 point on* $\tilde{Y}_a$*. Then, the quintic hypersurface* $\det M_q(z) = 0$ *is the trisecant variety* $\mathrm{Trisec}(\mathcal{Q}_a(Y))$*, and the quintic hypersurface* $\det L_q(z) = 0$ *is the secant variety* $\mathrm{Sec}(Y_a)$*.*

Finally, we summarise, using the Crauder-Katz Theorem (4.1), the described picture graphically as following diagram:

$$\begin{tikzcd}
 & & & \widehat{\mathbb{P}}^4_Y \arrow[dddll, dashed, "\sigma"'] & & & \\
 & & & \cup & & & \\
 & & & \widehat{\mathrm{Sec}}(Y) = \mathbb{P}^1_{S^2Y} \arrow[dl, dashed, "\sigma_Y"'] \arrow[dr] & & & \\
Y \arrow[r, hook] & \mathbb{P}^4(x) \arrow[dddrr, "\Phi"'] & \mathrm{Sec}(Y) \arrow[l, hook'] & & \mathrm{Trisec}(X) \arrow[r, hook] & \mathbb{P}^4(z) \arrow[uuull, "\Phi^{-1}"'] & X \simeq S^2Y \arrow[l, hook'] \\
 & & & \widehat{\mathrm{Trisec}}(X) = \mathbb{P}^2_Y \arrow[ul] \arrow[ur, dashed, "\sigma_X"'] & & & \\
 & & & \cap & & & \\
 & & & \widehat{\mathbb{P}}^4_X \arrow[uuurr, dashed, "\sigma"'] & & &
\end{tikzcd}$$

Here, the maps $\sigma : \widehat{\mathbb{P}}^4_Y \dashrightarrow \mathbb{P}^4(x)$ and $\sigma : \widehat{\mathbb{P}}^4_X \dashrightarrow \mathbb{P}^4(z)$ are blow-up maps along $Y \hookrightarrow \mathbb{P}^4(x)$ and along $X \hookrightarrow \mathbb{P}^4(z)$, respectively; the maps σ_Y and σ_X are blow-ups of the secant $\mathrm{Sec}(Y)$ and trisecant $\mathrm{Trisec}(X)$ varieties of Y and X and the maps Φ and Φ^{-1} are direct and inverse Cremona transfomations discussed above in Sect. 4, Theorem 4.1. We see that the map Φ is defined by quadrics through the elliptic quintic curve Y. The secant variety $\mathrm{Sec}(Y)$ is mapped by Φ to the quintic scroll $S^2(Y)$.

We remark that the degeneration locus of Φ maps to Trisec(X) the trisecant of $X = S^2(Y)$. In its turn, the inverse Cremona Φ^{-1} transfers this trisecant to Y and the degeneration locus maps to the secant variety Sec(Y). Coming back to the description of the the symplectic foliations for $q_{5,1}(Y)$ and $q_{5,2}(Y)$,we observe that the Cremona transformation and its inverse transform the symplectic leaves to the symplectic leaves.

The essential difference between the degeneration properties of these symplectic foliations was discussed by Pym (see [31]).

5.5 "Quantum" Cremona Transformations

Now, we are ready to explain the "quasi-classical" analogue of the following Feigin-Odesskii "quantum" statement (Theorems 4.1–4.3 in [9]):

Theorem 5.3.
- *The quantum Sklyanin algebra $Q_{5,2}(Y,\eta)$ with generators Z_i, $0 \le i \le 4$, and known 10 quadratic relations can be embedded into the quantum Sklyanin algebra $Q_{5,1}(Y,\eta)$ so that this homomorphism is given (on the level of generators) as $Z_i \mapsto q_i(\eta)$, where $q_i(\eta)$, $0 \le i \le 4$, are quadratic elements (so $Q_{5,2}(Y,\eta)$ is isomorphic to the subalgebra in $Q_{5,1}(Y,\eta)$ generated by $q_i(\eta)$). The elements $q_i(\eta) \to z_i = q_i$ (see 1) when $\eta \to 0$.*
- *The quantum Sklyanin algebra $Q_{5,1}(Y,\eta)$ with generators X_i, $0 \le i \le 4$ and known 10 quadratic relations, in its turn, can be embeded into the quantum Sklyanin algebra $Q_{5,2}(Y,\eta)$ so that this homomorphism is given (on the level of generators) as $X_i \mapsto \mathcal{Q}_i(\eta)$ where $\mathcal{Q}_i(\eta)$, $0 \le i \le 4$, are cubic elements (so $Q_{5,1}(Y,\eta)$ is isomorphic to the subalgebra in $Q_{5,2}(Y,\eta)$ generated by $Q_i(\eta)$). The elements $\mathcal{Q}_i(\eta) \to x_i = \mathcal{Q}_i$ (see 2) when $\eta \to 0$.*
- *The composition $Q_{5,1}(Y,\eta) \to Q_{5,2}(Y,\eta) \to Q_{5,1}(Y,\eta)$ maps the generators X_i, $0 \le i \le 4$, of $Q_{5,1}(Y,\eta)$ to the elements $K_{5,1}(\eta)X_i$, $0 \le i \le 4$, where $K_{5,1}(\eta) \to K_{5,1}(x)$ (16) when $\eta \to 0$.*
- *The composition $Q_{5,2}(Y,\eta) \to Q_{5,1}(Y,\eta) \to Q_{5,2}(Y,\eta)$ maps the generators Z_i, $0 \le i \le 4$ of $Q_{5,2}(Y,\eta)$ to the elements $K_{5,2}(\eta)Z_i$, $0 \le i \le 4$,*

  where $K_{5,2}(\eta) \to K_{5,2}(z)$ (17) when $\eta \to 0$.

The proof of this Theorem is a straightforward but lengthy computation.

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From Reflection Equation Algebra to Braided Yangians

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Abstract. In general, quantum matrix algebras are associated with a couple of compatible braidings. A particular example of such an algebra is the so-called Reflection Equation algebra In this paper we analyze its specific properties, which distinguish it from other quantum matrix algebras (in first turn, from the RTT one). Thus, we exhibit a specific form of the Cayley-Hamilton identity for its generating matrix, which in a limit turns into the Cayley-Hamilton identity for the generating matrix of the enveloping algebra $U(gl(m))$. Also, we consider some specific properties of the braided Yangians, recently introduced by the authors. In particular, we exhibit an analog of the Cayley-Hamilton identityfor the generating matrix of such a braided Yangian. Besides, by passing to a limit of this braided Yangian, we get a Lie algebra similar to that entering the construction of the rational Gaudin model. In its enveloping algebra we construct a Bethe subalgebra by the method due to D.Talalaev.

Keywords: Reflection equation algebra · Braided Lie algebra Affinization · Braided Yangian · Quantum symmetric polynomials Cayley-Hamilton identity

AMS Mathematics Subject Classification, 2010: 81R50

1 Introduction

Let V be a vector space, $\dim V = N$, and $R : V^{\otimes 2} \to V^{\otimes 2}$ be a braiding, i.e. a solution of the braid relation (also called the quantum Yang-Baxter equation)

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}, \qquad R_{12} = R \otimes I, \quad R_{23} = I \otimes R.$$

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Hereafter I stands for the identity operator or its matrix. The above relation is written in the space $V^{\otimes 3}$ and the lower indices label the spaces where a given operator acts.

The unital associative algebra generated by entries of a matrix $L = \|l_i^j\|_{1\le i,j\le N}$, subject to the following system

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R = 0, \quad L_1 = L \otimes I, \tag{1}$$

is called the *Reflection Equation* (RE) algebra, associated with a braiding R and denoted $\mathscr{L}(R)$. Below, the matrix L and all similar matrices are called *generating*.

The algebra $\mathscr{L}(R)$ is a particular case of the so-called quantum matrix (QM) algebras. Any QM algebra is associated with a couple of compatible braidings (see [12] for more details).

Another well-known example of a QM algebra is the so-called RTT algebra, generated by entries of a matrix $T = \|t_i^j\|_{1\le i,j\le N}$ subject to the system

$$R\,T_1\,T_2 - T_1\,T_2\,R = 0, \qquad T_1 = T \otimes I, \quad T_2 = I \otimes T. \tag{2}$$

All algebras, we are dealing with, are assumed to be unital.

A braiding R is called an *involutive symmetry*, if it meets the condition $R^2 = I$, and a *Hecke symmetry*, if it meets the Hecke relation[1]

$$(qI - R)(q^{-1}I + R) = 0, \qquad q \in \mathbb{K}, \quad q^2 \ne 1. \tag{3}$$

If an involutive or Hecke symmetry R is a deformation of the flip P, then the both QM algebras are deformations of the commutative algebra $\mathrm{Sym}(gl(N))$. This means that the dimensions of homogeneous components of each of these algebras are classical, i.e. equal to those of corresponding components[2] in $\mathrm{Sym}(gl(N))$. Emphasize that similar algebras can be associated with any braiding R but in general this deformation property fails. Below, all symmetries R which are deformations of the usual flips and the corresponding objects will be referred to as *deformation* ones.

The best known examples of deformation Hecke symmetries are those coming from the Quantum Groups (QG) $U_q(sl(N))$. However, we introduce all our algebras without any QG, which plays merely the role of a symmetry group for them, provided the corresponding Hecke symmetry R comes from this QG. As another example of a deformation Hecke symmetry we mention the Cremmer-Gervais R-matrices. However, in general the involutive and Hecke symmetries, we are dealing with, are not deformation either of the usual flips or of the super-ones.

Also, we assume all symmetries to be skew-invertible (see the next section for the definition). Under this condition a braided (or R-)analog of the trace can be defined. Note that this trace enters all our constructions. In particular, it takes part in the definition of quantum analogs of the symmetric polynomials in all

[1] From now on, the notation $\mathbb{K}$ stands for the ground field, which is $\mathbb{C}$ or $\mathbb{R}$.

[2] If R is a Hecke symmetry we should additionally require q to be generic, that is $q^n \ne 1$ for any integer n.

algebras under consideration. These quantum symmetric polynomials generate commutative subalgebras, called *characteristic*.

However, only in the RE algebras these subalgebras are central (see [10]). Besides, the RE algebras possess many other properties distinguishing them from other QM algebras. The main purpose of the present paper is to exhibit specific features of the RE algebras and of the so-called *braided Yangians* [6], which are current (i.e. depending on parameters) algebras in a sense close to the RE ones.

Here, we mention two of these particular properties. First, if a Hecke symmetry $R = R(q)$ is a deformation of the usual flip P (i.e. $R(1) = P$), then the corresponding *modified* RE algebra (16) is a deformation of the enveloping algebra $U(gl(N))$. It can be treated as the enveloping algebra of a *braided analog* $gl(V_R)$ of a Lie algebra $gl(n)$. If R is a skew-invertible Hecke symmetry of a general type, similar analogs of the Lie algebra $gl(N)$ and its enveloping algebra can be also defined.

In this connection the following question arises: whether it is possible to define an affine version of the braided Lie algebras similar to $\widehat{gl(N)}$? Below, we introduce such a braided analog $\widehat{gl(V_R)}$. Note that, putting aside the affine QG $U_q(\widehat{\gg})$, there are known two approaches to define quantum generalizations of affine algebras: the RE algebras in the spirit of [16][3] and the double Yangians and their q-analogs as introduced in [2]. In our subsequent publications we plan to study the centre of the enveloping algebra $U(\widehat{gl(V_R)})$ in the frameworks of the Kac's approach and to compare all mentioned methods of defining quantum affine algebras.

The second particular property of the RE algebra is that its generating matrix L satisfies a matrix polynomial identity $Q(L) = 0$ for a polynomial $Q(t)$, called *characteristic*. Thus, we get a version of the Cayley-Hamilton identity.

As was shown in [12], such an identity exists for the generating matrices of other QM algebras. However, only in the RE algebra this identity arises from the characteristic polynomial. Also, in deformation cases by passing to the limit $q \to 1$ in the modified form of the RE algebra, we get the characteristic polynomial and the corresponding Cayley-Hamilton identity for the generating matrix of the enveloping algebra $U(gl(N))$, which are usually obtained via the so-called Capelli determinant.

As noticed above, the braided Yangians,v recently introduced in [6], are in a sense close to the RE algebras. They are associated with current quantum R-matrices, constructed by means of the Yang-Baxterization of involutive and Hecke symmetries. These braided Yangians constitute one of two classes which generalize the Yangian $\mathbf{Y}(gl(N))$, introduced by Drinfeld [1]. The second class of Yangian-like algebras, also introduced in [6], consists of the so-called *Yangians of RTT type* which are more similar to the RTT algebras.

[3] They differ from our braided Yangians by the middle terms, which are also current R-matrices. Observe that there are known many versions of such RE algebras.

One of the main dissimilarities of the braided Yangians and these of RTT type arises from their evaluation morphisms. For the braided Yangians the evaluation morphisms are similar to these for the Yangians $\mathbf{Y}(gl(N))$, but their target algebras are the RE algebras (modified or not) instead of $U(gl(N))$. Another particular property of the braided Yangians is that they admit the Cayley-Hamilton identities for the generating matrices, which are also more similar to the classical ones. This is due to the fact that the analogs of the matrix powers entering these identities are given by the usual matrix product of several copies of the generating matrix (but with a shifted parameter u).

Also, deformation braided Yangians, in particular those, associated with R-matrices (35) and called *braided q-Yangians*, admit a limit[4] as $q \to 1$. In this limit we get Lie algebras $\mathfrak{G}_{trig}$ similar to $\mathfrak{G}$ entering construction of the rational Gaudin model. By using the method due to D.Talalaev, we construct Bethe subalgebras in the enveloping algebras $U(\mathfrak{G}_{trig})$. Consequently, we get new Bethe subalgebras in the Lie algebras $gl(N)^{\oplus K}$. In a more detailed way the corresponding version of an integrable model will be considered elsewhere.

The paper is organized as follows. In the next section we recall some basic properties of braidings and symmetries. In Sect. 3 we describe the RE algebra and the corresponding braided Lie algebra $gl(V_R)$. Also, we define its affinization. In Sect. 4 we consider different forms of the characteristic polynomials for the generating matrices of the RE algebras. In Sect. 5 we introduce braided Yangians and describe their specific properties. In the last section by passing to the $q = 1$ limit in the braided q-Yangian, we get the aforementioned current Lie algebra $\mathfrak{G}_{trig}$ and find a Bethe subalgebra in its enveloping algebra.

2 Braidings: Definitions and Properties

The starting object of our approach is a *skew-invertible* braiding of one of two types, specified below. Recall that a braiding R is called skew-invertible if there exists an operators $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that the following relation holds

$$\mathrm{Tr}_2 R_{12}\Psi_{23} = P_{13} \quad \Leftrightarrow \quad R_{ij}^{kl}\Psi_{lp}^{jq} = \delta_i^q\delta_p^k, \tag{4}$$

where the symbol Tr_2 means that the trace is applied in the second matrix space. Below a summation over repeated indexes is always understood. Here we assume that a basis $\{x_i\}$ in the space V is fixed and $\|R_{ij}^{kl}\|$ is the matrix of the operator R in the basis $\{x_i \otimes x_j\}$:

$$R(x_i \otimes x_j) = R_{ij}^{kl}\, x_k \otimes x_l.$$

The condition (4) enables us to extend R up to a braiding

$$R : V^{\otimes 2} \to V^{\otimes 2}, \quad (V^*)^{\otimes 2} \to (V^*)^{\otimes 2}, \quad V^* \otimes V \to V \otimes V^*, \quad V \otimes V^* \to V^* \otimes V,$$

[4] In order to get this limit we first change the basis in the Yangian, or in other words, we pass to the shifted form of this Yangian.

such that there exists an R-invariant pairing $V \otimes V^* \to \mathbb{K}$ (see [5]). This extended braiding implies a braiding

$$R^{\mathrm{End}(V)} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2},$$

where we identify $\mathrm{End}(V) \cong V \otimes V^*$ since V is a finite dimensional space.

Now, introduce two operators

$$B = \mathrm{Tr}_1 \Psi \quad \Leftrightarrow \quad B_i^j = \Psi_{ki}^{kj}, \qquad C = \mathrm{Tr}_2 \Psi \quad \Leftrightarrow \quad C_i^j = \Psi_{ik}^{jk}. \tag{5}$$

The definition of Ψ and the Yang-Baxter equation for R leads to the properties:

$$\mathrm{Tr}_1\, B_1 R_{12} = I_2, \qquad \mathrm{Tr}_2\, C_2 R_{12} = I_1. \tag{6}$$

$$R_{12} B_1 B_2 = B_1 B_2 R_{12}, \qquad R_{12} C_1 C_2 = C_1 C_2 R_{12}, \tag{7}$$

Let $\{x^j\}$ be the *right* dual basis of the space V^*, i.e. $< x_i, x^j >= \delta_i^j$. Then the R-invariant pairing in the opposite order is

$$<\,,\,>: V^* \otimes V \to \mathbb{K}, \quad < x^j, x_i >= B_i^j. \tag{8}$$

In the space $\mathrm{End}(V)$ we fix the following basis

$$l_i^j := x_i \otimes x^j \in \mathrm{End}(V)$$

and consider the map

$$\mathrm{tr}_R : \mathrm{End}(V) \to \mathbb{K}, \quad l_i^j \mapsto \delta_i^j, \tag{9}$$

motivated by the pairing $V \otimes V^* \to \mathbb{K}$. We call this map the R-trace.

Also, using the pairing (8), we define the following product in the space $\mathrm{End}(V)$

$$\circ : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V), \quad l_i^j \circ l_k^l = B_k^j\, l_i^l. \tag{10}$$

This product is $R^{\mathrm{End}(V)}$-invariant in the following sense

$$R^{\mathrm{End}(V)}(I \otimes \circ) = (\circ \otimes I) R_2^{\mathrm{End}(V)} R_1^{\mathrm{End}(V)}, \qquad R^{\mathrm{End}(V)}(\circ \otimes I) = (\circ \otimes I) R_1^{\mathrm{End}(V)} R_2^{\mathrm{End}(V)}, \tag{11}$$

where all operators act on the space $\mathrm{End}(V)^{\otimes 3}$. Hereafter, for the sake of simplicity we write R_k instead of $R_{k\,k+1}$.

Now, introduce the following pairing on the space $\mathrm{End}(V)$:

$$\langle\,,\,\rangle : \mathrm{End}(V)^{\otimes 2} \to \mathbb{K}, \quad \langle X, Y \rangle = \mathrm{tr}_R(X \circ Y), \quad \forall\, X, Y \in \mathrm{End}(V). \tag{12}$$

Thus, on the generators l_i^j we have $\langle l_i^j, l_k^l \rangle = B_k^j \delta_i^l$.

Now, let $M = \|m_i^j\|$ be an $N \times N$ matrix. Define its R-trace as

$$\mathrm{Tr}_R M = \mathrm{Tr}(CM) = M_i^j\, C_j^i.$$

This definition is motivated as follows. With a matrix M we associate an element $M_i^j x_j \otimes \tilde{x}^i \in \mathrm{End}(V)$, where $\tilde{x}^i$ is the *left* dual basis of the space V^*, i.e. $<\tilde{x}^i, x_j>= \delta_j^i$. The right R-invariant pairing of $\tilde{x}^i$ and x_j reads $< x_j, \tilde{x}^i >= C_j^i$ (see [5] for details). So, the R-trace of M is just the result of applying this pairing to $M_i^j x_j \otimes \tilde{x}^i$.

As was shown in [15], this R-trace has an important property: for any $N \times N$ matrix M the following holds

$$\mathrm{Tr}_{R(2)} R_1 M_1 R_1^{-1} = \mathrm{Tr}_{R(2)} R_1^{-1} M_1 R_1 = I_1 \mathrm{Tr}_R M.$$

From now on, we use the following notation $\mathrm{Tr}_{i_1 \dots i_k} = \mathrm{Tr}_{i_1} \dots \mathrm{Tr}_{i_k}$ where $i_1 < \dots < i_k$ and the same for R-traces.

Now, we assume R to be a Hecke symmetry and q to be generic. The corresponding constructions and results for involutive symmetries can be obtained by passing to the limit $q \to 1$.

Note that for any Hecke symmetry R the symmetric and skew-symmetric algebras

$$\mathrm{Sym}_R(V) = T(V)/\langle \mathrm{Im}(qI - R)\rangle, \qquad \textstyle\bigwedge_R(V) = T(V)/\langle \mathrm{Im}(q^{-1}I + R)\rangle$$

can be introduced. Since they are graded, the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k \dim \mathrm{Sym}_R^{(k)}(V) t^k, \qquad P_-(t) = \sum_k \dim \textstyle\bigwedge_R^{(k)}(V) t^k,$$

are well defined. Here the index (k) labels the k-th order homogeneous components. According to [9] the Poincaré-Hilbert series $P_\pm(t)$ are rational functions.

Emphasize that the above homogeneous components can also be defined via the projectors of symmetrization $\mathscr{P}_+$ (called below symmetrizers) and skew-symmetrization $\mathscr{P}_-$ (skew-symmetrizers)

$$\mathscr{P}_+^{(k)} : V^{\otimes k} \to \mathrm{Sym}_R^{(k)}(V), \qquad \mathscr{P}_-^{(k)} : V^{\otimes k} \to \textstyle\bigwedge_R^{(k)}(V).$$

The latter operators can be introduced by a recursive relation:

$$\mathscr{P}_-^{(k)} = \frac{1}{k_q}\, \mathscr{P}_-^{(k-1)} \left(q^{k-1} I - (k-1)_q R_{k-1}\right) \mathscr{P}_-^{(k-1)}, \qquad k_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad (13)$$

where we put by definition $\mathscr{P}_-^{(1)} = I$ and assume that the skew-symmetrizer $\mathscr{P}_-^{(k)}$ is always applied at the positions $1, 2, \dots, k$. Formula (13) was proved in [3] in a little bit different normalization of the Hecke symmetries.

Let us assume the rational function $P_-(t)$ to be noncancellable. Let m (respectively n) be the degree of its numerator (respectively, denominator). The ordered couple $(m|n)$ is called the *bi-rank* of the symmetry R.

If R is a skew-invertible symmetry (involutive or Hecke) and its by-rank is $(m|n)$ then the operators B and C (5) have a few additional properties:

$$BC = q^{2(n-m)} I, \qquad \mathrm{Tr}\, B = \mathrm{Tr}\, C = q^{n-m}(m-n)_q. \qquad (14)$$

Proposition 1. *If R is a skew-invertible Hecke symmetry and its bi-rank is $(m|0)$, then*

$$\mathrm{Tr}_{R(k+1\dots m)}\mathscr{P}_-^{(m)} = q^{-m(m-k)}\frac{k_q!(m-k)_q!}{m_q!}\mathscr{P}_-^{(k)}, \tag{15}$$

where we use the notation $k_q! = 1_q 2_q \dots k_q$ and the standard agreement $0_q! = 1$.

Proof. The proof of this claim is a direct consequence of recurrence (13) and properties of R-trace. Indeed, in virtue of the condition on the bi-rank and (14) we have

$$\mathrm{Tr}_R I = \mathrm{Tr}\, C = q^{-m}m_q,$$

while formula (6) means that $\mathrm{Tr}_{R(k)}R_{k-1} = I_{k-1}$. Now we can calculate a typical trace:

$$\mathrm{Tr}_{R(k)}\left(q^{k-1}I - (k-1)_q R_{k-1}\right) = \left(q^{k-m-1}m_q - (k-1)_q\right)I_{k-1} = q^{-m}(m-k+1)_q I_{k-1},$$

where at the last step we used the relation

$$q^a b_q - q^b a_q = (b-a)_q.$$

Thus, we have

$$\mathrm{Tr}_{R(m)}\left(q^{m-1}I - (m-1)_q R_{m-1}\right) = q^{-m}I_{m-1}$$

and consequently,

$$\mathrm{Tr}_{R(m)}\mathscr{P}_-^{(m)} = \frac{q^{-m}}{m_q}\mathscr{P}_-^{(m-1)}.$$

Upon applying the R-trace once more, we get

$$\mathrm{Tr}_{R(m-1,m)}\mathscr{P}_-^{(m)} = q^{-2m}\frac{2_q}{m_q(m-1)_q}\mathscr{P}_-^{(m-2)} = q^{-2m}\frac{2_q!(m-2)_q!}{m_q!}\mathscr{P}_-^{(m-2)}.$$

Now, using the reasoning by recursion, we arrive to formula (15). ∎

3 Braided Lie Algebras and their Affinization

Now, consider a unital associative algebra generated by matrix elements of $N\times N$ matrix $\tilde{L} = \|l_i^j\|$ which obey the system of quadratic-linear relations:

$$R\,\tilde{L}_1\,R\,\tilde{L}_1 - \tilde{L}_1\,R\,\tilde{L}_1\,R = R\,\tilde{L}_1 - \tilde{L}_1\,R. \tag{16}$$

We call this algebra the *modified RE algebra* and denote it $\tilde{\mathscr{L}}(R)$.

If $q^2 \neq 1$ the algebras $\mathscr{L}(R)$ (1) and $\tilde{\mathscr{L}}(R)$ are isomorphic to each other. The isomorphism is realized by the following map

$$\tilde{L} \mapsto L + \frac{1}{q-q^{-1}}I. \tag{17}$$

Due to this reason we treat the algebra $\tilde{\mathscr{L}}(R)$ as a modified form of the algebra $\mathscr{L}(R)$. In [5] we have constructed a representation category of the algebra $\tilde{\mathscr{L}}(R)$ similar to that of the algebra $U(gl(N))$. The isomorphism (17) enables us to convert any $\tilde{\mathscr{L}}(R)$-module into a $\mathscr{L}(R)$-one.

Here, we want to mention only three $\tilde{\mathscr{L}}(R)$-modules. The first one is the basic space V. The corresponding *vector representation* is defined by

$$\rho_V(l_i^j) \triangleright x_k = B_k^j x_i.$$

where the notation $\triangleright$ stands for the action of a linear operator.

The second $\tilde{\mathscr{L}}(R)$-module, called *covector*, is defined in the dual space V^* by the following action on basis elements

$$\rho_{V^*}(l_i^j) \triangleright x^k = -x^l R_{li}^{kj}.$$

The third module, called *adjoint*, is identified with $V \otimes V^*$. The action of the elements l_i^j on this module is defined by means of the following coproduct

$$\Delta(l_i^j) = l_i^j \otimes 1 + 1 \otimes l_i^j - (q - q^{-1}) \sum_k l_i^k \otimes l_k^j. \tag{18}$$

Onto the whole algebra $\tilde{\mathscr{L}}(R)$ this coproduct is extended by means of the braiding $R^{\mathrm{End}(V)}$. In this sense we speak about a *braided bi-algebra structure* of the algebra $\tilde{\mathscr{L}}(R)$. The reader is referred to [5] for details. Note, that the coproduct (18) arises from the braided structure of the RE algebra discovered by Sh. Majid [13].

Another way to define the adjoint representation is based on a *braided analog* of the Lie bracket. It is defined as follows. The system quadratic-linear relations (16) on the generators of the algebra $\tilde{\mathscr{L}}(R)$ can be rewritten as

$$l_i^j \otimes l_k^l - \mathscr{R}(l_i^j \otimes l_k^l) = [l_i^j, l_k^l], \tag{19}$$

where

$$\mathscr{R} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2} \quad \text{and} \quad [\,,\,] : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)$$

are two operators. The operator $\mathscr{R}$ is defined below (see (25)). The operator $[\,,\,]$ is by definition the result of applying the operator (10) to the left hand side of (19).

Emphasize that if $q = 1$ the operators $\mathscr{R}$ and $R^{\mathrm{End}(V)}$ coincide with each other but for a generic q it is not so.

Then the adjoint representation of the algebra $\tilde{\mathscr{L}}(R)$ can be defined as followed

$$\rho_{\mathrm{End}(V)}(l_i^j) \triangleright l_k^l = [l_i^j, l_k^l]. \tag{20}$$

Proposition 2. *The action (20) defines a representation of the algebra $\tilde{\mathscr{L}}(R)$.*

In order to prove this claim it suffices to show that the action (20) coincides with that discussed above. It can be also shown by straightforward computations.

Definition 1. The space $\mathrm{End}(V)$ equipped with the operators $\mathscr{R}$ and $[\,,\,]$ is called braided Lie algebra and is denoted $gl(V_R)$.

Also, the algebra $\tilde{\mathscr{L}}(R)$ plays the role of the enveloping algebra of the braided Lie algebra $gl(V_R)$ in virtue of Proposition 2.

Besides the property formulated in Proposition 2, the braided Lie algebra $gl(V_R)$ has the following features.

1. Its bracket $[\,,\,]$ is skew-symmetric in the following sense: $[\,,\,]\mathscr{P} = 0$. Here $\mathscr{P} : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V)^{\otimes 2}$ is a symmetrizer, constructed in [5]. Note that in comparison with the above projectors $\mathscr{P}_{\pm}$ acting in tensor powers of the space V, the symmetrizer $\mathscr{P}$ acts in $(\mathrm{span}_{\mathbb{K}}(l_i^j))^{\otimes 2}$. Such symmetrizers and analogical skew-symmetrizers were constructed in [5] for the tensor powers 2 and 3 of the space $\mathrm{span}_{\mathbb{K}}(l_i^j)$.

2. This bracket is $R^{\mathrm{End}(V)}$-invariant in the same sense as in (11). In the case related to the QG $U_q(sl(N))$, this bracket is also covariant with respect to the action of this QG.

As for (20), we treat it as a braided analog of the Jacobi identity. Note that if R is an involutive symmetry, then the corresponding Jacobi identity can be written under the following form, similar to the classical (or super-)one:

$$[\,,\,][\,,\,]_{12}(I + R_1^{\mathrm{End}(V)} R_2^{\mathrm{End}(V)} + R_2^{\mathrm{End}(V)} R_1^{\mathrm{End}(V)}) = 0. \tag{21}$$

Also, note that the right hand side of (16) can be obtained by applying the product $\circ$ (10) to its left hand side. This follows from the relation

$$\circ L_1 R_{12} L_1 = L_1 \mathrm{Tr}_{(1)} B_1 R_{12} = L_1 I_2.$$

Now, introduce the following useful notation. We put[5]

$$L_{\overline{1}} = L_1, \quad L_{\overline{k}} = R_{k-1} L_{\overline{k-1}} R_{k-1}^{-1}, \quad k \ge 2. \tag{22}$$

With the use of this notation, we can rewrite the system (1) in a form similar to (2):

$$R_1 L_{\overline{1}} L_{\overline{2}} - L_{\overline{1}} L_{\overline{2}} R_1 = 0. \tag{23}$$

The adjoint action can also be written as

$$L_{\overline{1}} \triangleright R_1\, L_{\overline{1}} = L_{\overline{1}} - R_1^{-1}\, L_{\overline{1}}\, R_1 \quad \Leftrightarrow \quad L_{\overline{1}} \triangleright L_{\overline{2}} = L_{\overline{1}}\, R_1^{-1} - R_1^{-1}\, L_{\overline{1}} = L_{\overline{1}}\, R_1 - R_1\, L_{\overline{1}}, \tag{24}$$

where in the last equality we use the following consequence of Hecke condition (3):

$$R^{-1} = R - (q - q^{-1})I.$$

[5] Note that the QM algebras as introduced in [12], are defined in a similar way, but with the help of the second braiding F, in a sense compatible with R: $L_{\overline{k}} = F_{k-1} L_{\overline{k-1}} F_{k-1}^{-1}$.

As for the operators $\mathscr{R}$ and $R^{\mathrm{End}(V)}$, they can be respectively presented as

$$\mathscr{R}(L_{\overline{1}}\otimes L_{\overline{2}}) = R^{-1}(L_{\overline{1}}\otimes L_{\overline{2}})R, \qquad R^{\mathrm{End}(V)}(L_{\overline{1}}\otimes L_{\overline{2}}) = L_{\overline{2}}\otimes L_{\overline{1}}. \tag{25}$$

Below, we use similar notations for dealing with the so-called braided Yangians.

Now, consider the element

$$\ell = \mathrm{Tr}_R L = \mathrm{Tr}(CL) \in \mathrm{End}(V).$$

In order to stress the difference between the R-traces Tr_R and tr_R (see (9)) note that $\mathrm{tr}_R L = I$. We have

$$\mathrm{tr}_R \ell = C_k^k = \frac{(m-n)_q}{q^{m-n}}.$$

Let us suppose that $m \neq n$ and consequently $tr_R \ell \neq 0$. Then the elements

$$f_i^j = l_i^j - \delta_i^j \frac{\ell}{\mathrm{tr}_R \ell}$$

are well defined and traceless: $\mathrm{tr}_R f_i^j = 0$. This enables us to define a braided analog $sl(V_R)$ of the Lie algebra $sl(N)$. The reader is referred to [5] for details.

Now, consider the affinization procedure of the braided Lie algebras $gl(V_R)$. For the algebras $sl(V_R)$ it can be done in a similar manner. Following the classical pattern, we consider the algebra

$$gl(V_R)[t,t^{-1}] = gl(V_R)\otimes \mathbb{K}[t,t^{-1}].$$

This algebra is generated by elements $l_i^j[a] := l_i^j \otimes t^a$, $a \in \mathbb{Z}$. The braided Lie bracket in it is also defined according to the classical pattern:

$$[X[a], Y[b]] := [X,Y][a+b], \quad X, Y \in gl(V_R).$$

To construct the central extension of $gl(V_R)[t,t^{-1}]$ we introduce a vector space

$$gl(V_R)[t,t^{-1}] \oplus \mathbb{K}\, c$$

where c is a new generator commuting with any elements of $gl(V_R)[t,t^{-1}]$. Besides, we extend the action of the operator $\mathscr{R}$ in a natural way

$$\mathscr{R}(X[a]\otimes Y[b]) = \mathscr{R}(X\otimes Y)[b][a], \quad \mathscr{R}(X[a]\otimes c) = c\otimes X[a], \quad \mathscr{R}(c\otimes X[a]) = X[a]\otimes c.$$

Here, the notation $\mathscr{R}(X\otimes Y)[b][a]$ means that we attribute the label b to the first factor and that a to the second one.

Now, define the *affine braided Lie algebra* $\widehat{gl(V_R)}$ by introducing the following bracket

$$[X[a], c] = 0, \quad [X[a], Y[b]] = [X,Y][a+b] + \omega(X[a], Y[b])\, c,$$

where

$$\omega(X[a], Y[b]) := a\langle X, Y\rangle \delta(a+b).$$

Here $\langle\, ,\, \rangle$ is the pairing (12) in the algebra $gl(V_R)$ and a discrete δ-function $\delta(a)$ is defined in the standard way:

$$\delta(a) = \begin{cases} 1 \text{ if } a = 0 \\ 0 \text{ if } a \neq 0. \end{cases}$$

We do not know what is the Jacobi identity in the braided Lie algebra $\widehat{gl(V_R)}$, provided R is a Hecke symmetry. However, if R is an involutive symmetry, the corresponding Jacobi identity is similar to (21). This claim can be easily deduced from the following property of the term ω.

Proposition 3. *If R is an involutive symmetry, then the following holds*

$$\omega\, [\, ,\,]_{23} \Big((I + R_1^{\mathrm{End}(V)} R_2^{\mathrm{End}(V)} + R_2^{\mathrm{End}(V)} R_1^{\mathrm{End}(V)})(X[a] \otimes Y[b] \otimes Z[c]) \Big) = 0.$$

In virtue of this property the term ω can be called *braided cocycle*.

The enveloping algebra $U(\widehat{gl(V_R)})$ can be also defined in a natural way as the quotient of the free tensor algebra of $\widehat{gl(V_R)}$ over the ideal, generated by the elements

$$c\, l_i^j[a] - l_i^j[a]\, c, \quad X[a]Y[b] - \mathscr{R}(X[a] \otimes Y[b]) - [X[a], Y[b]] - \omega(X[a], Y[b])c.$$

In a similar manner it is possible to define the enveloping algebra $U(sl(V_R))$. In fact, we suggest a new way of introducing quantum analogs of affine Lie algebras.

In our subsequent publications we plan to study the center of the algebras $U(gl(V_R))$ and $U(sl(V_R))$ in the spirit of the Kac's approach.

4 Characteristic Polynomials for Generating Matrices

In this section we suppose that the bi-rank of a given skew-invertible Hecke symmetry R is $(m|0)$, $m \geq 2$. As was shown in [4], the generating matrix L of the corresponding RE algebra meets the following Cayley-Hamilton identity

$$L^m - qL^{m-1}e_1(L) + q^2 L^{m-2} e_2(L) + \ldots + (-q)^{m-1} L e_{m-1}(L) + (-q)^m I e_m(L) = 0, \tag{26}$$

where

$$e_0(L) = 1, \qquad e_k(L) := Tr_{R(1\ldots k)}(\mathscr{P}_-^{(k)} L_{\overline{1}} L_{\overline{2}} \ldots L_{\overline{k}}), \quad k \geq 1,$$

are quantum analogs of the elementary symmetric polynomials. Here $\mathscr{P}_-^{(k)}$: $V^{\otimes k} \to V^{\otimes k}$ is the skew-symmetrizer (13).

Note that quantum analogs of these and other symmetric polynomials (Schur polynomials, power sums) are also well defined in all QM algebras and they generate a commutative subalgebra called *characteristic*. As we said above, in the RE algebra the characteristic subalgebra is central. By this reason, it does not matter on what side of the powers of the matrix L we put the coefficients in the Cayley-Hamilton identity and in its generalizations called Cayley-Hamilton-Newton identities. An important consequence of this fact is the possibility to introduce the quantum spectrum of L, the quantum eigenvalues μ_i of L belong to an algebraic extension of the center of RE algebra. This quantities allows one to rewrite the Cayley-Hamilton identity (26) in a factorized form:

$$\prod_{i=1}^{N}(L - \mu_i I) = 0.$$

By contrary, in other QM algebras the elements e_k are not central and their position in the Cayley-Hamilton identity (in front of a matrix power or behind it) is important. Besides, in these algebras the powers L^k should be replaced by their quantum counterparts, which also exist in two forms. We refer the reader to the paper [12], where the Cayley-Hamilton identity (and its generalization called the Cayley-Hamilton-Newton identity) is proved for the generating matrices of all QM algebras, the RTT and RE algebras included.

Remark 1. In [11] the notion of a half-quantum algebra was introduced. Similarly to a QM algebra a half-quantum algebra is defined with the help of a couple (R, F) of compatible braidings, but the defining relations on generators are less restrictive than those for the QM algebras. The point is that analogs of symmetric polynomials can be also defined in half-quantum algebras and a version of the Cayley-Hamilton-Newton relations can be established. Nevertheless, in general, the "symmetric polynomials" in these algebras do not commute with each other.

As follows from formula (26), the characteristic polynomial for the generating matrix L of the RE algebra is

$$Q(t) = \sum_{k=0}^{m} t^{m-k}(-q)^k e_k(L), \tag{27}$$

since $Q(t)$ is the m-th order polynomial with the unit coefficient at the highest power t^m which possesses the property $Q(L) \equiv 0$.

Our current aim is to present this polynomial in a form useful for finding the characteristic polynomial for the generating matrix of the modified RE algebra. By passing to the limit $q \to 1$, we get a characteristic polynomial for the generating matrix of the algebra $U(gl(N))$.

Proposition 4. *The polynomial $Q(t)$ defined in (27) is identically equal to the expression:*

$$Q(t) = q^m \,\mathrm{Tr}_{R(1\dots m)} \left(\mathscr{P}_-^{(m)}(tI - L_{\overline{1}})(q^2 tI - L_{\overline{2}})\dots(q^{2(m-1)}tI - L_{\overline{m}}) \right). \tag{28}$$

Proof. Consider the following polynomial in m indeterminates t_i:

$$\hat{Q}(t_1, \dots, t_m) = q^m \,\mathrm{Tr}_{R(1\dots m)} \left(\mathscr{P}_-^{(m)}(t_1 I - L_{\overline{1}})(t_2 I - L_{\overline{2}})\dots(t_m I - L_{\overline{m}}) \right).$$

By developing the product of linear factors in the above expression, we get a sum with the typical term

$$q^m\sigma_k(t_1,\dots,t_m)\,\mathrm{Tr}_{R(1\dots m)}\left(\mathscr{P}_-^{(m)}(-L_{\overline{1}})\dots(-L_{\overline{m-k}})\right),$$

where $\sigma_k(t_1,\dots,t_m)$ are the elementary symmetric polynomials in $t_1,...,t_m$:

$$\sigma_k(t_1,\dots,t_m)=\sum_{1\le i_1<\dots<i_k\le m} t_{i_1}\dots t_{i_k}.$$

Here, we use an essential fact that the factors $L_{\overline{i}}$ *under the R-trace* can be "shifted" to the most possible left position. This means that the following identities hold

$$\mathrm{Tr}_{R(1\dots m)}\left(\mathscr{P}_-^{(m)}L_{\overline{s}_1}L_{\overline{s}_2}\dots L_{\overline{s}_k}\right)=\mathrm{Tr}_{R(1\dots m)}\left(\mathscr{P}_-^{(m)}L_{\overline{1}}L_{\overline{2}}\dots L_{\overline{k}}\right)$$

for any set of integers $1\le s_1<s_2<\dots<s_k\le m$. Emphasize that this property is specific for the RE algebra. It fails if the matrices $L_{\overline{k}}$ are defined via a braiding F different from R (see footnote 5).

By using Proposition 1, we can present the polynomial $\hat{Q}$ as follows

$$\hat{Q}(t_1,\dots,t_m)=\sum_{k=0}^{m}(-1)^k q^{-m(m-k-1)}\,\frac{k_q!(m-k)_q!}{m_q!}\,\sigma_{m-k}(t_1,\dots,t_m)\,e_k(L). \tag{29}$$

Now, we take into account a result from q-combinatorics:

$$\sigma_k(t,q^2t,...,q^{2(k-1)}t)=t^k\sigma_k(1,q^2,...,q^{2(k-1)})=q^{k(m-1)}\frac{m_q!}{k_q!(m-k)_q!}\,t^k.$$

So, specializing $t_k=q^{2(k-1)}t$ in the above expression (29) we precisely get the formula (27). Therefore

$$Q(t)=\hat{Q}(t,q^2t,\dots,q^{2(m-1)}t),$$

which proves the claim in virtue of the the definition of $\hat{Q}$. ∎

It is obvious, that formula (28) can be written as follows:

$$Q(t)=q^m\,\mathrm{Tr}_{R(1\dots m)}\left(\mathscr{P}_-^{(m)}(tI-L)_{\overline{1}}(q^2tI-L)_{\overline{2}}\dots(q^{2(m-1)}tI-L)_{\overline{m}}\right).$$

From this form of the characteristic polynomial (28) for the matrix L we can get the characteristic polynomial for the matrix $\tilde{L}$.

Corollary 1. *The characteristic polynomial $\tilde{Q}(t)$ for the matrix $\tilde{L}$ is equal to*

$$\tilde{Q}(t)=q^m\,\mathrm{Tr}_{R(1\dots m)}\left(\mathscr{P}_-^{(m)}\prod_{k=1}^{m}\left(q^{2(k-1)}\left(t-q^{-k+1}(k-1)_q\right)I-\tilde{L}_{\overline{k}}\right)\right), \tag{30}$$

where the factors are placed in ascending order in k from the left to right.

Proof. The generating matrices L and $\tilde{L}$ are connected by a linear shift (17). Introducing a new indeterminate

$$\tilde{t} = t + \frac{1}{q - q^{-1}},$$

we obviously have $tI - L = \tilde{t}I - \tilde{L}$. Therefore, rewriting L and t in (28) in terms of $\tilde{L}$ and $\tilde{t}$ we get the polynomial $\tilde{Q}(\tilde{t})$ with the property $\tilde{Q}(\tilde{L}) = 0$ (we return to the notation t instead of $\tilde{t}$ at the end of transformations). ∎

Note, that the value m in the above formulae is in general independent of the parameter $N = \dim V$ except for the restriction $m \le N$. If a given Hecke symmetry is a deformation of the usual flip (the $U_q(sl(N))$ Drinfeld-Jimbo R-matrix as a well-known example), then its bi-rank is $(N|0)$. By passing to the limit $q \to 1$ we get the following claim.

Corollary 2. *Let $M = \|m_i^j\|$ be the generating matrix of the enveloping algebra $U(gl(N))$, where m_i^j $1 \le i, j \le N$ is the standard basis of $gl(N)$. Then the following polynomial is characteristic for this matrix*

$$\mathscr{Q}(t) = \mathrm{Tr}_{(1\dots N)}\left(\mathscr{P}_-^{(N)}(tI - M_1)((t-1)I - M_2)\dots((t-N+1)I - M_N)\right),$$

where $\mathscr{P}_-^{(N)}$ is the usual skew-symmetrizer in $V^{\otimes N}$, that is we have $\mathscr{Q}(M) = 0$.

Emphasize that usually the characteristic polynomial for the matrix M is constructed by means of the Capelli determinant.

Also, note that the claim of the Corollary 1 is valid for the generating matrix of any modified RE algebra corresponding to a skew-invertible involutive symmetry R, provided R can be approximated by a family of Hecke symmetries.

Remark 2. Along with the characteristic polynomial for the generating matrix of the enveloping algebra $U(gl(N))$ one usually exhibits a similar polynomial for its transposed matrix M^t (see [14]). In our setting the matrix M^t should be replaced by the generating matrix of the modified RE algebra defined by

$$R_1L_2R_1L_2 - L_2R_1L_2R_1 = R_1L_2 - L_2R_1. \tag{31}$$

It is treated as the enveloping algebra of the algebra of right endomorphisms of V. For the generating matrix of this algebra the characteristic polynomial is similar to (28) but the R-trace should be defined via $\mathrm{Tr}_R L = \mathrm{Tr}(BL)$ and all the matrices $L_{\overline{k}}$ should be replaced by $L_{\underline{k}}$ where

$$L_{\underline{m}} = L_m, \quad L_{\underline{k}} = R_k^{-1} L_{\underline{k+1}} R_k, \ 1 \le k \le m-1.$$

5 Braided Yangians

Let $R(u,v)$ be a current quantum R-matrix. This means that it is subject to the quantum Yang-Baxter equation

$$R_{12}(u,v)R_{23}(u,w)R_{12}(v,w) = R_{23}(v,w)R_{12}(u,w)R_{23}(u,v).$$

Consider an analog of the RTT algebra, associated with such an R-matrix defined by the system

$$R(u,v)L_1(u)L_2(v) = L_1(v)L_2(u)R(u,v), \tag{32}$$

where the matrix

$$L(u) = \sum_{k\geq 0} L[k]u^{-k}, \qquad L[k] = \|l_i^j[k]\|_{1\leq i,j\leq N}, \tag{33}$$

expands in a series in non-positive powers of the parameter.

Thus, the system (32), being rewritten via the generators $l_i^j[k]$, leads to an infinite family of quadratic equations on the generators $l_i^j[k]$, whose number is also infinite, but each equation is a polynomial in the generators.

The R-matrix $R(u,v)$ we are dealing with are of two classes:

$$R(u,v) = R - \frac{a}{u-v}I, \qquad R(u,v) = R - \frac{(q-q^{-1})u}{u-v}. \tag{34}$$

In the first formula in (34) R stands for a skew-invertible involutive symmetry, whereas in the latter one $R = R(q)$ is a skew-invertible Hecke symmetry. The fact that these R-matrices meet the quantum Yang-Baxter equation can be verified by a straightforward calculation [6]. The procedure of constructing such current R-matrices via symmetries R is often called *Yang-Baxterization.*

Note that the Drinfeld's Yangian corresponds to the famous Yang R-matrix, which is defined via the first formula (34) with $R = P$.

Observe that the second current R-matrix in (34) is actually depending on the ratio u/v of the parameters. There exists another (so-called *trigonometrical*) form which can be obtained via the change of variables $u \mapsto q^u$, $v \mapsto q^v$. After such a transformation the current R-matrix will depend on the difference $u - v$. Below, we deal with the rational form (34) of the current R-matrix.

All Yangian-like algebras defined via (32) but with other current R-matrices are called *Yangians of RTT type.*

The well-known examples of Yangians of RTT type are the so-called q-Yangians (see [14]). In the lowest dimensional case such a q-Yangian is associated with the following current R-matrix

$$R(u,v) = R - \frac{(q-q^{-1})u}{u-v}I = \begin{pmatrix} \frac{-qv+q^{-1}u}{u-v} & 0 & 0 & 0 \\ 0 & \frac{(-q+q^{-1})v}{u-v} & 1 & 0 \\ 0 & 1 & \frac{(-q+q^{-1})u}{u-v} & 0 \\ 0 & 0 & 0 & \frac{-qv+q^{-1}u}{u-v} \end{pmatrix}. \tag{35}$$

Note that each Yangian of RTT type has a bi-algebra structure. On the generators the corresponding coproduct is defined as follows

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(l_i^j(u)) = l_i^k(u) \otimes l_k^j(u).$$

Now, consider the so-called *evaluation morphism*

$$T(u) \to T + \frac{\overline{T}}{u}.$$

This map induces a morphism of algebras, if R is a Hecke symmetry and the matrices T and $\overline{T}$ meet the following relations

$$R\,T_1\,T_2 = T_1\,T_2\,R, \quad R\,\overline{T}_1\,\overline{T}_2 = \overline{T}_1\,\overline{T}_2\,R, \quad R\,\overline{T}_1\,T_2 = T_1\,\overline{T}_2\,R.$$

Thus, the target algebra generated by the entries of the matrices T and $\overline{T}$ is a couple of RTT algebras connected by the last relation.

Note that in the case of q-Yangians one usually imposes some additional conditions on the matrix $L[0]$.

In [6] we suggested another candidate for the role of the q-Yangian. Consider an algebra generated by entries of a matrix $L(u)$ subject to the relations

$$R(u,v)L_1(u)RL_1(v) - L_1(v)RL_1(u)R(u,v) = 0, \tag{36}$$

where R is just the involutive or Hecke symmetry entering the current R-matrix (34). Besides, in the expansion (33) we assume that $L[0] = I$. We denote this algebra $\mathbf{Y}(R)$ and call it the *braided Yangian*.

Below, we use the notation similar to (22):

$$L_{\overline{1}}(u) = L_1(u), \quad L_{\overline{k}}(u) = R_{k-1}L_{\overline{k-1}}(u)R_{k-1}^{-1}, \quad k \geq 2.$$

By using this notation, it is possible to cast the defining relations (36) in a form similar to the Yangians of RTT type

$$R(u,v)L_{\overline{1}}(u)L_{\overline{2}}(v) - L_{\overline{1}}(v)L_{\overline{2}}(u)R(u,v) = 0.$$

The braided Yangian, corresponding to the R matrix (35) and its higher dimensional analogs are called *braided q-Yangian*.

Let us mention some of properties of the braided Yangians. First of all, any braided Yangian has a braided bi-algebra structure. The corresponding coproduct is defined on the generators by the same formulae as in the Yangians of RTT type but it is extended on the whole algebra via the braiding $R^{\mathrm{End}(V)}$ in a way similar to that in the RE algebra.

Another important property of the braided Yangians is that their evaluation morphisms look like the classical one

$$L(u) \to I + \frac{M}{u}. \tag{37}$$

The target algebra generated by entries of the matrix M, is described by the following claim proved in [6].

Proposition 5. *1. If R is an involutive symmetry, then the map (37) defines a surjective morphism $\mathbf{Y}(R) \to \tilde{\mathscr{L}}(R)$. Besides, the map $M \mapsto L[1]$ defines an injective morphism $\tilde{\mathscr{L}}(R) \to \mathbf{Y}(R)$.*
2. If R is a Hecke symmetry, then the map (37) defines a morphism $\mathbf{Y}(R) \to \mathscr{L}(R)$.

Thus, the type of the target algebra depends on the type of the initial symmetry R. This proposition enables us to construct a large representation category of each braided Yangian. We describe it for the braided q-Yangian $\mathbf{Y}(R)$.

Consider the category of finite dimensional $U_q(sl(N))$-modules which are deformations of the $U(sl(N))$-ones. Each of its objects can be endowed with a structure of the $\tilde{\mathscr{L}}(R)$-module where R is coming from the QG $U_q(sl(N))$. This fact follows from the method of constructing the category of $\tilde{\mathscr{L}}(R)$-module as was done in [4]. Finally, by using the isomorphism (17) we can convert any $\tilde{\mathscr{L}}(R)$-module into $\mathscr{L}(R)$-one. Now, it remains to apply the above proposition.

One of the most remarkable properties of the Yangians of all types is that analogs of some symmetric functions are well defined in these algebras. Also, analogs of the Cayley-Hamilton-Newton identities are valid in these algebras. Note that these identities can be presented in different form. Below, we exhibit them in a form which differs from that of [6,7].

First, define analogs of powers $L^k(u)$ and skew-powers $L^{\wedge k}(u)$ of the matrix $L(u)$, generating a braided Yangian:

$$L^k(u) := L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)\dots L(u),$$
$$L^{\wedge k}(u) := \mathrm{Tr}_{R(2\dots k)}\Big(\mathscr{P}_-^{(k)} L_{\overline{1}}(u)L_{\overline{2}}(q^{-2}u)\dots L_{\overline{k}}(q^{-2(k-1)}u)\Big) \quad k \geq 2,$$

where it is convenient to set by definition $L^0(u) = I$ and $L^{\wedge 1}(u) = L(u)$.

Here, we would like to emphasize a difference between the braided Yangians and these of RTT type. In the latter ones analogs of the matrix powers and skew-powers can be also defined, but only in the braided Yangians analogs of the matrix powers are defined via the usual matrix product of the generating matrices (but with shifted parameters).

In the braided Yangians quantum analogs of the power sums and elementary symmetric polynomials are respectively defined by

$$p_k(u) = \mathrm{Tr}_R L^k(u) = \mathrm{Tr}_R L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)\dots L(u),$$
$$e_k(u) = \mathrm{Tr}_R L^{\wedge k}(u) = \mathrm{Tr}_{R(1\dots k)}\Big(\mathscr{P}_-^{(k)} L_{\overline{1}}(u)L_{\overline{2}}(q^{-2}u)\dots L_{\overline{k}}(q^{-2(k-1)}u)\Big). \tag{38}$$

Now, we are able to exhibit the Cayley-Hamilton-Newton identities in the braided Yangians.

Proposition 6. *The following matrix identities hold true for the generating matrix of a braided Yangian*

$$(-1)^{k+1} k_q L^{\wedge k}(u) = \sum_{p=1}^{k} (-q)^{k-p} L^p(q^{-2(k-p)}u)e_{k-p}(u) \quad \forall\, k \geq 1. \tag{39}$$

Proof. Can be done by the considerations similar to these from [6], namely, by sequential use of the recurrent formula (13). However, first, we present the skew-power $L^{\wedge k}(u)$ as follows

$$L^{\wedge k}(u) = \mathrm{Tr}_{R(2\dots k)}\Big(L_{\overline{1}}(q^{-2(k-1)}u)L_{\overline{2}}(q^{-2(k-2)}u)\dots L_{\overline{k}}(u)\mathscr{P}_-^{(k)}\Big).$$

Then we apply the method used in [6].

Note, that in the cited paper we got a different form of the Cayley-Hamilton-Newton identities: the elementary symmetric functions appeared there on the *left* of matrix powers of L and the matrix powers were defined by more complicated expressions than those written above. ■

If the bi-rank of the Hecke symmetry R is $(m|0)$ (in particular, for the braided q-Yangian it is $(N|0)$, i.e. $m = N$), then the highest nonzero skew-power is

$$L^{\wedge m}(u) = q^m e_m(u)\, I,$$

where $e_m(u)$ is the highest nonzero elementary symmetric polynomial, which is a quantum analog of the determinant.

Consequently, on setting in (39) $k = m$ we get the Cayley-Hamilton identity

$$\sum_{p=0}^{m}(-q)^p L^{m-p}(q^{-2p}u)e_p(u) = 0.$$

By applying the R-trace to the identities (39), we get a family of the quantum Newton identities

$$p_k(u) - qp_{k-1}(q^{-2}u)e_1(u) + (-q)^2 p_{k-2}(q^{-4}u)e_2(u) + \dots \\ + (-q)^{k-1}p_1(q^{-2(k-1)}u)e_k(u) + (-q)^k k_q e_k(u) = 0 \quad \forall\, k \geq 1.$$

Emphasize that the quantum analog of the determinant $e_m(u)$ is central in the braided Yangian $\mathbf{Y}(R)$ for any symmetry R, whereas in the Yangians of RTT type its centrality depends on R (see [6]). The other analogs of elementary symmetric polynomials $e_k(u)$, $1 \leq k \leq m$ and power sums $p_k(u)$, $k \geq 1$, commute with each other and generate a commutative *Bethe subalgebra*. This is true also for the Yangians of RTT type.

6 Shifted Braided q-Yangian and its $q = 1$ Limit

Let $r(u, v)$ be a classical r-matrix, i.e. $gl(N)$-valued function in parameters u and v (assumed to be rational), which meets the classical Yang-Baxter equation

$$[r_{12}(u,v), r_{13}(u,w)] + [r_{12}(u,v), r_{23}(v,w)] + [r_{13}(u,w), r_{23}(v,w)] = 0.$$

Suppose that the map

$$L(u) \otimes L(v) \mapsto \{L_1(u), L_2(v)\} = [r(u,v), L_1(u) + L_2(v)], \qquad (40)$$

is skew-symmetric and consequently it defines a Poisson bracket. Also, suppose that the matrix-function $L(u) = \|l_i^j(u)\|$ expands in a series (33). Thus, this Poisson bracket, defined on the commutative algebra $\mathrm{Sym}(gl(N)[t^{-1}])$, can be expressed via the coefficients $L[k]$, $k \geq 0$.

By using the standard R-matrix technique, it is easy to show that the elements $\mathrm{Tr}L^k(u)$ commute with each other with respect to this Poisson bracket:

$$\{\mathrm{Tr}L^k(u), \mathrm{Tr}L^l(v)\} = 0, \quad \forall\, k, l \geq 0 \tag{41}$$

for any values u and v.

The simplest non-constant example of an r-matrix is the following one

$$r(u,v) = \frac{P}{u-v}. \tag{42}$$

The corresponding Poisson bracket is an important ingredient of the rational Gaudin model and its quantization.

Let us emphasize that the elements $\mathrm{Tr}L^k(u)$ (or their R-counterparts) do not commute any more in the enveloping algebra $U(\mathfrak{G})$ of the Lie algebra $\mathfrak{G}$ defined by formula (40) with the same r-matrix (42). Talalaev [17] succeeded in finding a Bethe subalgebra in this enveloping algebra.

Now, consider the map

$$L(u) \mapsto \sum_{k=1}^{K} \frac{M_k}{u-u_k}, \tag{43}$$

where $u_k \in \mathbb{K}$ are K fixed points, and the matrices M_k generate K copies of the algebra $U(gl(N))$. This means that each matrix M_k generates the enveloping algebras $gl(N)$ and entries of any two matrices of this family commute with each other. Then, the map (43) defines a Lie algebra morphism $\mathfrak{G} \to \mathfrak{g} = gl(N)^{\oplus K}$ and consequently a morphism of the enveloping algebras of these Lie algebras.

In fact, the map (43) can be treated as an analog of the evaluation morphism $\mathbf{Y}(gl(N)) \to U(gl(N))$, combined with the morphisms $u \mapsto u - u_k$ and the usual coproduct.

The image of the Bethe subalgebra in the algebra $U(\mathfrak{G})$ under the map (43) is a Bethe subalgebra in the algebra $U(\mathfrak{g})$. Some quadratic elements of the latter algebra play the role of the Hamiltonians of the rational Gaudin model. Talalaev's result gives rise to higher Hamiltonians of this model.

Let us emphasize that constructing a Bethe subalgebra in the algebra $U(\mathfrak{G})$ was performed by Talalaev via using a Bethe subalgebra in the Yangian $Y(gl(N))$. It is tempting to replace the r-matrix (42) by that corresponding to R-matrix (35) and upon using the same method, to find a Bethe subalgebra in the enveloping algebra of the corresponding Lie algebra.

Unfortunately, this method fails though a Bethe subalgebra in the q-Yangian of RTT type exists and is known (see [6]). This failure is due to the fact that in the q-Yangian of RTT type it is not possible to find elements of this Bethe subalgebra which have a controllable expansion in the deformation parameter h.

We claim that the Talalaev's method is still valid in the braided q-Yangian. Namely, below we exhibit elements of the Bethe subalgebra of this generalized Yangian which has the necessary expansion property. This enables us to find a Bethe subalgebra in the enveloping algebra $U(\mathfrak{G}_{trig})$ of the Lie algebra $\mathfrak{G}_{trig}$, which arises (similarly to the rational case) from linear Poisson structure corresponding to this braided q-Yangian (see [6]). Besides, an analog of the map (43) can be also found and used for constructing a Bethe subalgebra in the algebra $U(\mathfrak{g})$.

As claimed in [6,8], the quantum elementary symmetric polynomials $e_k(u)$ defined by (38) commute with each other in the braided Yangians $\mathbf{Y}(R)$. Also, we consider the following elements

$$\hat{e}_k(u) = \mathrm{Tr}_{R(1\dots m)}\Big(\mathscr{P}_-^{(m)} L_{\overline{1}}(u) L_{\overline{2}}(q^{-2}u)\dots L_{\overline{k}}(q^{-2(k-1)}u)\Big),$$

which differ from $e_k(u)$ by a modification of the skew-symmetrizers and the positions where the R-traces are applied. According to Proposition 1 any element $\hat{e}_k(u)$ differs from that $e_k(u)$ by a non-trivial (for a generic q) numerical factor. Consequently, the elements $\hat{e}_k(u)$ also commute with each other.

By using the relation

$$q^{-2\partial_u} f(u) = f(q^{-2}u) q^{-2\partial_u}, \ \text{ where } \ \partial_u = u\frac{d}{du},$$

we can present the elements $\hat{e}_k(q^{-2}u)$ as follows

$$\hat{e}_k(q^{-2}u) = \mathrm{Tr}_{R(1\dots m)}\Big(\mathscr{P}_-^{(m)} (q^{-2\partial_u} L_{\overline{1}}(u))(q^{-2\partial_u} L_{\overline{2}}(u))\dots(q^{-2\partial_u} L_{\overline{k}}(u))\Big) q^{2k\partial_u}. \tag{44}$$

Now, change the basis of the braided q-Yangian in a way similar to (17):

$$L(u) = (q - q^{-1})\tilde{L}(u) + I. \tag{45}$$

Also, we put $q^2 = \exp(h)$ and expand the elements $\hat{e}(q^{-2}u)$, expressed via the matrix $\tilde{L}$, in h.

Since the factors entering formula (44) expand as

$$q^{-2\partial_u} L_{\overline{p}}(u) = (1 - h\partial_u + o(h))(I + h\tilde{L}_{\overline{p}}(u) + o(h)) = I + h(\tilde{L}_{\overline{p}}(u) - I\,\partial_u) + o(h),$$

we get the following expansion of the elements $\hat{e}_k(q^{-2}u)$

$$\hat{e}_k(e^{-h}u) = \mathrm{Tr}_{R(1\dots m)}\Big(\mathscr{P}_-^{(m)}(I+h(\tilde{L}_{\overline{1}}(u)-I\,\partial_u)+o(h))\dots(I+h(\tilde{L}_{\overline{k}}(u)-I\,\partial_u)+o(h))\Big)(1+o(1)).$$

Also, note that $\tilde{L}_{\overline{p}}(u) = \tilde{L}_p(u) + o(1)$ for all p.

Following [17], consider the elements

$$\tau_k(u) = \sum_{p=0}^{k} (-1)^{k-p}\frac{k!}{p!(k-p)!}\hat{e}_p(u),$$

commuting with each other in the braided q-Yangian.

We state that the expansion of the element $\tau_k(e^{-h}u)$ begins with a term proportional to h^k. Thus, the elements $h^{-k}\tau_k(e^{-h}u)$ expand as follows

$$h^{-k}\tau_k(e^{-h}u) = QH_k(u) + o(h).$$

This entails that the elements $QH_k(u), k = 0, 1..., m$ commute with each other in the algebra which is the limit of the braided q-Yangian as $h \to 0$. Let us compute the defining relations of the limit algebra. Being expressed via the matrix $\tilde{L}$ the defining system of the braided q-Yangian is

$$(R - \frac{(q-q^{-1})u}{u-v} I)\tilde{L}(u)_1 R\tilde{L}(v)_1 - \tilde{L}(v)_1 R\tilde{L}(v)_1 (R - \frac{(q-q^{-1})u}{u-v} I) = -[R, \frac{u\tilde{L}_1(u) - v\tilde{L}_1(v)}{u-v}].$$

Consequently, the defining relations of the limit algebra are

$$[\tilde{L}_1(u), \tilde{L}_2(v)] = [P, \frac{u\tilde{L}_1(u) - v\tilde{L}_1(v)}{u-v}] = [\frac{P}{u-v}, u\tilde{L}_1(u) + v\tilde{L}_2(v)]. \tag{46}$$

We denote $\mathfrak{G}_{trig}$ the Lie algebra with the bracket, defined by the right hand side of this formula. Thus, in the basis $\tilde{L}(u)$ the braided q-Yangian turns into the enveloping algebra $U(\mathfrak{G}_{trig})$ of this Lie algebra as $q \to 1$ (or $h \to 0$).

Thus, similarly to [17] we have the following.

Proposition 7. *The elements*

$$QH_k(u) = \mathrm{Tr}_{(1\dots m)}\mathscr{P}_-^{(m)}(\tilde{L}_1(u) - I\partial_u)(\tilde{L}_2(u) - I\partial_u)...(\tilde{L}_k(u) - I\partial_u)1, \ \ 1 \geq k \geq m$$

commute with each other in the algebra $U(\mathfrak{G}_{trig})$.

Observe that here the trace and skew-symmetrizer are classical.

The commutative subalgebra generated by the elements $QH_k(u)$ in the algebra $U(\mathfrak{G}_{trig})$ is called the *Bethe subalgebra*.

Remark 3. Note that similarly to (17) the map (45) converts a quadratic algebra in a quadratic-linear one. However, the defining relations of the limit algebra do not depend on the concrete form of the Hecke symmetry R.

Now, describe an analog of the map (43).

Proposition 8. *The map*

$$\tilde{L}(u) \mapsto \sum_{k=1}^{K} \frac{M_k u_k}{u - u_k},$$

where the family $(M_1, ..., M_K)$ generates the Lie algebra $\mathfrak{g} = gl(m)^{\oplus K}$ (in the same sense as in (43)), defines a Lie algebra morphism $\mathfrak{G}_{trig} \to \mathfrak{g}$ and consequently, a morphism of the enveloping algebras of these Lie algebras.

The corresponding version of an integrable model will be considered elsewhere.

Let us point out two main differences of our result from that of [17]. First, our ∂_u is the "multiplicative derivative": $u\frac{d}{du}$. Second, the Lie algebra $\mathfrak{G}_{trig}$ cannot be presented under the form (40). The Poisson structure, corresponding to this Poisson bracket, is exhibited in [7]. Also, a comparative analysis of the Poisson structures related to different types of the deformation Yangians is presented there. Emphasize that the Poisson structures corresponding to the braided Yangians do not also depend on the concrete form of the Hecke symmetry as well. Up to a factor, it equals the right hand side of formula (46).

Completing the paper, emphasize once more that the Talalaev's method is still valid since we are dealing with the braided version of the Yangians.

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Survey of the Deformation Quantization of Commutative Families

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Abstract. In this survey chapter we discuss various approaches and known results, concerning the following question: when is it possible to find a commutative extension of a Poisson-commutative subalgebra in $C^\infty(X)$ (where X is a Poisson manifold) to a commutative subalgebra in the deformation quantization of X, the algebra $\mathscr{A}(X)$. A case of particular interest, which we consider with certain detail is the situation, when $X = \mathfrak{g}^*$ and the commutative subalgebra is constructed by the argument shift method.

Keywords: Deformation quantization
Poisson commutative subalgefbras · (Quantum) integrable systems

1 Introduction

1.1 The Main Question

Let X be a Poisson manifold, i.e. a closed compact manifold with a fixed bivector $\pi \in \Lambda^2 TX$, satisfying the relation

$$[\pi, \pi] = 0,$$

where $[,]$ is the Schouten bracket of polyvector fields. Equivalently, one can say, that there is a Poisson bracket $\{f, g\} = \pi(df, dg)$ on the algebra of smooth functions on X, satisfying the Leibniz rule in both variables and the Jacobi identity. An important particular case of Poisson manifolds is when π is a nondegenerate bilinear form on T^*X at all points. In this case its inverse $\omega = \pi^{-1} \in \Lambda^2 T^*X$ is a symplectic structure on X.

It is known, that every Poisson manifold allows *deformation quantization*, i.e. that there exists an $\hbar$-linear associative noncommutative $*$-product on $C^\infty(X)[[\hbar]]$, given by formula

$$f * g = fg + \frac{\hbar}{2}\{f, g\} + \sum_{k=2}^{\infty} \hbar^n B_n(f, g), \tag{1}$$

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where $B_n,\ n \geq 2$ are certain differential operators in both variables (this is sometimes called *bidifferential* operators); we shall usually set $B_1(f,g) = \frac{1}{2}\{f,g\}$. For example, one can take B_n from Kontsevich's formula (see [3]):

$$B_n(f,g) = \sum_{\Gamma \in G_n} w_\Gamma B_\Gamma(f,g),$$

where $\Gamma \in G_n$ is certain set of graphs, B_Γ is a differential operator, associated with every graph, and π and w_Γ is a weight, given by an integral over a configuration space. Below we shall often abbreviate the algebra $(C^\infty(X)[[\hbar]],*)$ simply to $\mathscr{A}(X)$. It is easy to see, that in case $\pi = 0$ this formula does not change the product at all. On the other hand, if $X = \mathbb{R}^{2n},\ \pi = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \cdots + \frac{\partial}{\partial p_n} \wedge \frac{\partial}{\partial q_n}$ (i.e. if the Poisson structure on $\mathbb{R}^{2n}$ is in effect constant and symplectic), then the algebra $\mathscr{A}(X)$ is isomorphic to the Weyl algebra of (formal polynomial) differential operators on $\mathbb{R}^{2n}$.

Now the main problem, that we try to solve, is: *given a commutative Poisson subalgebra $C \in C^\infty(X)$, is it possible to find a commutative algebra $\mathscr{C}$ in $\mathscr{A}(X)$, quantizing C*, i.e. such that there is a linear isomorphism $f \mapsto \hat{f} \in \mathscr{C} \subseteq \mathscr{A}(X)$, where

$$\hat{f} = f + \sum_n \hbar^n f_n,\ \ f_n \in C^\infty(X),$$

and $\hat{f} * \hat{g} = \hat{g} * \hat{f}$ for all $f, g \in C$.

In many situations such commutative algebras have important applications. For instance, in case of the constant symplectic structure on $\mathbb{R}^{2n}$, we obtain commutative subalgebras inside the Weyl algebras, whose structure is closely related to such important problem as the Jacobian problem (see [1]). Commutative subalgebras in the rings of differential operators and their generalizations (e.g.in universal enveloping algebras) have been extensively studied in many situations, especially under the name of *quantum integrable systems*, see for instance [2].

Besides this, one can use the obstructions, that arise in the process of quantization as a method to distinguish different nonequivalent Poisson structures on manifolds: these obstructions are invariants of the Poisson structure (under the morphisms, preserving the commutative subalgebra), so if they vanish for one Poisson structure and are nontrivial for another, it would mean that these two structures cannot be identified by a Poisson morphism.

In this survey we shall give an account of the known results in this field, as well as of the methods and ideas, used in the investigations. We try to make our exposition self-contained, so that an interested reader will be (at least) given references to the papers, in which the corresponding results have been published.

1.2 Argument Shift Method and Enveloping Algebras

One of the main subject of our interest is the commutative Poisson subalgebras, generated by the *argument shift method* (principal details of this method are explained in the Sect. 2.1). The advantage of these algebras is that the corresponding construction is purely algebraic, so there are all chances that these algebras can in fact be quantized.

There are certain evidences, that it is in fact the case. First of all, if the manifold X is given by a Lie algebra, i.e. in the original case, where the argument shift method first appeared, the deformed algebra is closely related to the universal enveloping algebra of $\mathfrak{g}$, i.e. $\mathscr{A}(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$, as linear spaces where

$$U(\mathfrak{g}) = T^{\otimes}\mathfrak{g}/\langle X \otimes Y - Y \otimes X - [X, Y]\rangle.$$

The relation, loosely speaking, is determined by setting $\hbar = 1$ in $\mathscr{A}(\mathfrak{g})$, and using the PBW theorem to identify $U(\mathfrak{g})$ with $S(\mathfrak{g})$ as linear spaces.

It has been a long standing problem to describe the commutative subalgebras in the universal enveloping algebras. One of the first known results here are the Harish-Chandra and Duflo isomorphism theorems, which identify the center of $U(\mathfrak{g})$ with the Weil-invariant part of $S(\mathfrak{h})$ (here $\mathfrak{h}$ is the Cartan subalgebra) and with $S(\mathfrak{g})^G$ (the Ad_G-invariants in $S(\mathfrak{g})$) respectively.

These results were followed by the works of Vinberg and Tarasov in 1990s (see [4,5]), in which other commutative subalgebras in the ubiversal enveloping algebras were introduced. Namely, in the latter case it was shown that the standard symmetrization map

$$S(\mathfrak{g}) \ni x_1 \dots x_k \mapsto \sum_{\sigma \in S_k} \frac{1}{k!} x_{\sigma(1)} \cdots x_{\sigma(k)} \in U(\mathfrak{g}),$$

where $x_i \in \mathfrak{g}$ and $x \cdot y$ denotes the product in $U(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ sends the Poisson commutative algebra of Mishchenko and Fomenko to a commutative subalgebra of $U(\mathfrak{g})$.

Later these results were further generalized by various people, so that now one knows that a somewhat similar statement holds for arbitrary semisimple Lie algebra. Namely, in Rybnikov's paper [6] the corresponding commutative subalgebras were described. This construction, however, based on the active use of the Kac algebras representation theory, is much less explicit. The existence of the necessary commutative subalgebras follows from the description of the center of the universal enveloping algebra of Kac algebra, $U(\hat{\mathfrak{g}})$ on critical level. It is shown, that there exists a homomorphism from $U(\hat{\mathfrak{g}})$ to $U(\mathfrak{g})$, which maps this center into a commutative subalgebra in $U(\mathfrak{g})$; moreover, one can choose certain elements in the center $Z(U(\hat{\mathfrak{g}}))$, such that the leading terms of their images in the enveloping algebra will coincide with the algebra of shifted invariants of Mishchenko and Fomenko.

Unfortunately, this construction is very indirect in that the construction of the corresponding central elements depends very much on the properties of the representation category of $U(\hat{\mathfrak{g}})$ at the critical level and it is not always possible to compute the explicit formulas for the corresponding elements. Some work in that direction has been done with the help of the Yangians (see for instance [2,7]), so that in many particular cases such formulas were eventually found. However, it is still not clear, in what proportion can this theory be transfered to non-semisimple algebras, e.g. on nilpotent algebras.

1.3 Presentation and Organization of the Text

This text is divided into two main parts. The first of it is dedicated to the description of the main properties of the argument shift method in Poisson algebras and its generalizations, so that the algebraic properties of this construction and its relation to the Nijenhuis operators are revealed. In Sect. 2.1, we describe the canonical construction, that goes back to Manakov [8], Mishchenko and Fomenko (see [9], though we use a somewhat more generic approach, so that it would include not only Lie algebras, but more general Poisson manifolds). In Sects. 2.2 and 2.3 a still more algebraic point of view is adapted, that relates this construction with the classical notion of Nijenhujs operators (see also [10], where a very close approach is used, known as the Lenard-Magri chains).

In the second part we describe the homological constructions that lurk beyond the deformation quantization. We begin with the classical result of Kontsevich, his "formality theorem", which relates the star-product with L_∞-maps between two given differential Lie algebras (Sect. 3.1). After this in Sect. 3.2 we give a brief account of results by Garay and van Straten (see [13]), who gave a convenient cohomological criterion for the quantizability of a commutative subalgebra. The following two Sects. 3.3 and 3.4 describe similar constructions from other authors (see [14,15]), which give homological obstruction for quantization of commutative subalgebras and Poisson vector field actions; the latter question is closely related to the problem we consider, since one can quantize the commutative subalgebra of Hamiltonian vector fields, generated by the integrable system as an approximation to the quantization of the system. In the last section, Sect. 3.5 of this part we give few examples of applications of the methods from previous sections.

Finally, in the Sect. 4 we give few remarks, concerning possible future investigations and pose several related problems, which can be of some value for the main one. The work of G.S. was supported by the Russian Science Foundation grant project No. 16-11-10069. The work of A.K. was supported by the Russian Science Foundation grant project No.17-11-01303.

2 The Argument Shift Method and Nijenhuis Operators

2.1 Argument Shift on a Manifold

The argument shift method is a rather simple, purely algebraic and comparatively universal way to generate commutative subalgebras in the Lie algebras of functions on a Poisson manifold X. It was first introduced by Manakov and Mischenko, Fomenko in 1970s (see [8,9]) in the case when $X = \mathfrak{g}^*$: the main result of their work is that for any two Ad^*_G-invariant functions f, g on $\mathfrak{g}^*$, and any vector ξ in $\mathfrak{g}^*$, iterated directional derivatives of f and g will commute:

$$\left\{\frac{\partial^k f}{\partial \xi^k}, \frac{\partial^l g}{\partial \xi^l}\right\} = 0,$$

for all k, l.

It turns out, that this construction can be easily generalized to a wide class of manifolds. Namely, let (X, π) be a Poisson manifold; one says that a vector field $\xi \in Vect(X)$ is *Nijenhuis field*, if $L_\xi^2\pi = 0$. In the terms of Schouten bracket on X, this equality is equivalent to

$$[[\pi, \xi], \xi] = 0.$$

It turns out, that in this case the Lie derivative π_ξ of π is again a Poisson bivector; we shall denote the bracket with respect to π_ξ by $\{,\}_\xi$.

Let now f and g be two Casimir functions on (X, π) (i.e. the functions, whose Hamilton vector fields with respect to π vanish identically; the set of such functions is sometimes called "the center of the Poisson algebra" and denoted $Z_\pi(C^\infty(X))$). Then one has a similar equation:

$$\{L_\xi^k(f), L_\xi^l(g)\} = 0,$$

for all k, l (here $L_\xi(f)$ denotes the pairing of vector field with a function, i.e. it is just $\xi(f)$). The proof is rather simple; for instance, one can reason by induction: we start with the equality $\{f, g\} = 0$, then, by differentiating it along ξ we obtain

$$\{f, g\}_\xi = -\{L_\xi(f), g\} - \{f, L_\xi(g)\} = 0,$$

which holds, since both functions are Casimirs. Differentiating this equality along ξ one more time and using the fact that ξ commutes with $L_\xi\pi$, and hence it is a differentiation of $\{,\}_\xi$, we obtain

$$\{L_\xi(f), g\}_\xi + \{f, L_\xi(g)\}_\xi = 0.$$

Similarly, differentiating $\{L_\xi(f), g\} = 0 = \{f, L_\xi(g)\}$, we obtain

$$\begin{aligned}\{L_\xi(f), g\}_\xi &= -\{L_\xi^2(f), g\} - \{L_\xi(f), L_\xi(g)\} = -\{L_\xi(f), L_\xi(g)\},\\ \{f, L_\xi(g)\}_\xi &= -\{L_\xi(f), L_\xi(g)\} - \{f, L_\xi^2(g)\} = -\{L_\xi(f), L_\xi(g)\},\end{aligned}$$

since f and g are Casimirs. Thus, we obtain

$$\{L_\xi(f), L_\xi(g)\} = \{L_\xi(f), g\}_\xi = \{f, L_\xi(g)\}_\xi = 0.$$

Repeating similar computations with these equations (and with $\{f, L_\xi^2(g)\} = 0 = \{L_\xi^2(f), g\}$), we obtain $\{L_\xi^2(f), L_\xi(g)\} = 0 = \{L_\xi(f), L_\xi^2(g)\}$ and so on.

It is important to observe, that the relations that we obtain in this way do not, in fact, depend on the Leibniz rule, verified by ξ and by the Poisson bracket. Moreover, the first step of this reasoning (i.e. that $\{\xi(f), \xi(g)\} = 0$) can be proved even if the right hand side of equation $[[\pi, \xi], \xi] = 0$ is a nontrivial bidifferential operator $P(f, g)$, with only one condition, that it should vanish for f, g from the center of our Poisson algebra. For instance, one can take $P(f, g) = \{Q_1(f), g\}$ or $P(f, g) = Q_2(\{f, g\})$, where Q_1, Q_2 are linear differential operators on X, or even take P equal to a linear combination of such operators: they all vanish, if both functions are Casimirs.

The algebraic properties of this construction allows one transfer it easily to the quantized algebra. In fact, replacing the Poisson brackets in the relations above by the commutator in $\mathscr{A}(X)$, we see that it is enough to find an operator $\hat{\xi}$ on $\mathscr{A}(X)$ such that

$$\hat{\xi} = \xi + \hbar\xi_1 + \dots,$$

which would verify the relation $L^2_{\hat{\xi}}([,]) = 0$ (where $[,]$ is the commutator of elements with respect to the star-product). In this case the same reasoning will yield the formula $[\hat{\xi}(f), \hat{\xi}(g)] = 0$ for any f, g from the center of $\mathscr{A}(X)$. Moreover, it can be shown that there exists a straightforward way to obtain elements from the center of $\mathscr{A}(X)$ from the center of Poisson algebra (see Sect. 3.1), so that for $f \in Z_\pi(C^\infty(X))$, the corresponding element in $\mathscr{A}(X)$ will have the form $\hat{f} = f + \hbar f_1 + \dots$. Thus the application of $\hat{\xi}$ will yield us a quantization of the algebra of shifted Casimirs.

2.2 Algebra Behind Argument Shift Method

In this section, we further explore the algebra behind the argument shift method. As we explained above, this method originally works with Lie algebras, but it can be generalized onto variety of cases rather easy.

Let V be a vector space (in general infinite dimensional). Consider the vector spaces C^i of i-linear mappings $\underbrace{V \times \dots \times V}_{\text{i times}} \to V$. For the sake of simplicity we assume that C^0 is the original vector space. In case of finite dimensional vector field C^i is just simply the space of tensors of type $(1, i)$

Consider now the graded vector space $\mathfrak{C} = C^0 \oplus C^1 \oplus \dots$. One can introduce an algebraic structure on $\mathfrak{C}$ by the following construction, very close to the Gerstenhaber bracket (see Sect. 3.1). Namely, we obtain a multiplication as the following combination of compositions: consider $v \in C^i, w \in C^j$; then

$$v \circ w = \sum_{k=1}^{i} v(\dots, \underbrace{w(\dots)}_{\text{kth place}}, \dots).$$

What we do here, is basically we take v, substitute w into different places and then take the sum of all the $i + j - 1$-linear operations we get. This operation has an interesting property: let $A(v, u, w)$ denote the *associator* of u, v, w with respect to this operation, that is

$$A(v, w, u) = v \circ (w \circ u) - (v \circ w) \circ u.$$

The name is justified by the simple observation, that an operation $\circ$ is associative, iff its associator vanishes identically. In our case it is not true, but the following lemma holds

Lemma 1. *For the operation $\circ$, introduced above the associator is left-symmetric, that is the following identity*

$$A(v, w, u) = A(w, v, u)$$

holds for arbitrary triple v, u, w.

This lemma can be proved by a direct computation. The algebras, for which associator has this property, are called *left-symmetric algebras*. So $\mathfrak{C}$, equipped with $\circ$ is left-symmetric algebra. An important property of left-symmetric algebras is described by the following

Lemma 2. *Let $\mathfrak{C}$ be an arbitrary left-symmetric algebra with operation $\circ$. Then $[v, w] = v \circ w - w \circ v$ defines a Lie algebra structure on $\mathfrak{C}$.*

It is easy to see that the only property one might need to check is the Jacobi identity. It turns out, that it follows from the left symmetry of the associator. That is why such algebras are also sometimes called *pre-Lie algebras*. Note also, that C^1 is closed under this commutator and is simply the Lie algebra of operators from $V \to V$ with standard commutator. Clearly, this construction resembles that of the Gerstenhaber bracket Lie algebra, but it differs from that one because there is no sign rule holding for the grading here.

The pre-Lie algebra $\mathfrak{C}, [\,,\,]$ is the main tool to define argument shift method. We shall call *argument shift triple* the following three objects: $V, R \in C^1, B \in C^2$ together with the property $[[B, R], R] = 0$.

Example 1. In original argument shift method V is $C^\infty(\mathfrak{g}^*)$ or $S(\mathfrak{g})$ with $\mathfrak{g}$ being Lie algebra. This space is infinite dimensional. Operator R is the derivative ∂_a, where a is an arbitrary element from $\mathfrak{g}^*$ and B is the standard Poisson Lie bracket.

Example 2. In argument shift method version on a Poisson manifold X, one should take $V = C^\infty(X)$. This space in also infinite dimensional. Operator R is the Nijenhuis vector field on this manifold and B is the Poisson bracket on this manifold.

The following theorem provides the basis for argument shift method to be of a use in different applications (it is a generalization of the reasoning from the previous section).

Theorem 1. *Consider an argument shift triple (V, B, R). Let $f, g \in V$ be from left and right kernels of B (this is a bilinear operation, not symmetric in general, so the kernels may differ) respectively. Then for arbitrary integers m, n we have:*

$$B(R^n f, R^m g) = 0.$$

The proof of this theorem requires some work in terms of properties of commutator. In case B is skew-symmetric as in the Poisson case, the left and right kernels coincide, so $R^n f, R^m g$ commute with respect to B.

Note that there exists a similar generalization of the notion of Nijenhuis tensor for arbitrary bilinear operation. Namely, we shall call *Nijenhuis triple* the following three objects: vector space V, operator $R \in C^1$ and a bilinear operation $B \in C^2$ such, that

$$[[B, R], R] = [B, R^2], \tag{2}$$

where R^2 is square of operator, that is $R \circ R$ in our notions.

It is remarkable, that a similar theorem holds for Nijenhuis triples:

Theorem 2. *Consider Nijenhuis triple* (V, B, R)*. Let* $f, g \in V$ *be from left and right kernels of* B *respectively (this bilinear operation is not symmetric in general, so the kernels may differ). Then for arbitrary integers* m, n *we have:*

$$B(R^n f, R^m g) = 0.$$

Again in Poisson case, the left and right kernels coincide, so $R^n f, R^m g$ commute with respect to B for arbitrary f, g in its center.

2.3 Algebraic Nijenhuis Operator and Argument Shift Method

Let us make few remarks, concerning the properties, which the the argument shift method has in that particular case, when the triple we consider comes from a Lie algebra.

More accurately, consider the following Nijenhuis triple (V, B, R), where V is a finite-dimensional vector space, B is some commutator map $[\,,\,]$, i.e. any bilinear map which is skew-symmetric and satisfies Jacobi identity (so that V possesses a Lie algebra structure) and R is a linear operator with the property, discussed in Sect. 2.2. Note, that throughout this section $[\,,\,]$ stands for a Lie algebra structure on V, and not the one on $\mathfrak{C}$, which we discussed above.

The property (2) in our case can be written as follows:

$$R[Rx, y] + R[x, Ry] - R^2[x, y] - [Rx, Ry] = 0,$$

where $x, y \in V$. Note, that it looks exactly the same as the definition of the Nijenhuis torsion on a manifold, with brackets having different meaning.

It is a simple exercise to show, that any such operator R on a Lie algebra can be easily extended to a left-invariant operator field on the corresponding Lie group using left multiplication by group elements (the same way left-invariant metrics are constructed from scalar products on Lie algebras). One readily sees that this left-invariant operator field R has vanishing Nijenhuis torsion.

Algebraic Nijenhuis operators on V can be used to construct commutative sets in $S(V)$. The method (which is a generalization of the construction in the previous section) goes as follows. Let

$$[x, y]_R = [Rx, y] + [x, Ry] - R[x, y].$$

It follows from (2) that this operation satisfies the Jacobi identity. Using this operation one may rewrite the Nijenhuis property in the following way:

$$R([x, y]_R) = [Rx, Ry]. \tag{3}$$

This means that R can be interpreted as a homomorphism of Lie algebras (from V with the usual Lie bracket to the same space, but with the bracket $[\,,\,]_R$). One can extend the homomorphism R to a homomorphism of algebras $S(V) \to S(V)$, intertwining the Lie-Poisson brackets introduced above.

Observe now that if R satisfies Nijenhuis identity, then $1+\lambda R$, where λ is a real parameter, also satisfies this identity.

Let us use the following notation:

$$[x, y]_\lambda = [x, y]_R - \lambda[x, y].$$

If we set $R_\lambda = R - \lambda E$ then we have the following identity, generalizing the Eq. (3):

$$R_\lambda([x, y]_\lambda) = [R_\lambda x, R_\lambda y].$$

This linear family of Lie structures determines a linear family of Poisson structures (Poisson pencil) $\{\, ,\, \}_\lambda$. Note, that in general there is no Nijenhuis triple here, as operator R itself is not extendable (or at least there exists no evident way to do so) to a differential operator on $S(V)$ so that Nijenhuis property is preserved. It is an interesting question: if a triple, associated with such pencil exists or not?

The following theorem is an analog of the argument shift method in the case we describe here.

Theorem 3. *Denote by V^λ the set of Casimir functions of $\{,\}_\lambda$. Then the union of V^λ for all λ not in spectrum of the operator R (the original one) is commutative with respect to the entire pencil.*

Example 3. Let V be $\mathfrak{gl}(n)$ and $R(X) = AX$ for a fixed matrix A, that is R is the operator of left multiplication by A. Then this is a Nijenhuis triple. The Poisson commutative subalgebra that one gets using the previous theorem coincides with the subalgebra, generated by argument shift method. Note, that this is the pencil, studied by Manakov in his oiriginal work on argument shift method.

Example 4. Let V be $\mathfrak{so}(n)$ and $R(X) = AX + XA$, where A is symmetric matrix. This is not Nijenhuis triple, but the Poisson pencil generated by this operator is the same as in previous example. However although $\mathfrak{so}(n)$ is not invariant under left multiplication by A, the functions obtained in this way are commutative as functions on this algebra.

On the other hand, it turns out that for every argument shift method triple on $S(V)$ one can associate an algebraic Neijenhuis triple in the following way. Consider the direct sum $V^2 = V \oplus V$, and introduce the Lie algebra on it so that the first copy of V is given by the usual Lie algebra structure and the second copy of V is in the center. That is we consider a central extension of the original Lie algebra.

Denote by e_i and f_j the basis in both components. We introduce the operator R on V^2 by setting $R(e_i) = f_i,\ R(f_i) = 0$. It is easy to check that $N_R = 0$. This gives us a Nijenhuis triple on V^2. To get argument shift method subalgebra one needs to use the natural projection $S(V \oplus V) \to S(V)$ called the evaluation morphism, i.e. it is given by substituting instead of f_i the numbers a_i.

3 Cohomological Constructions

3.1 Hochschild Cohomology and Kontsevich Formality Theorem

After Kontsevich formality theorem was proved, one of the main ideas of the deformation theory is to exploit its relation with Hochschild (co)homology; of course, this relation has been known before, but its importance manifested itself best of all in Kontsevich's result. Let us begin with a brief account of the corresponding construction.

Let A be an associative algebra, e.g. $A = C^\infty(M)$ for a smooth manifold M. Then the Hochschild cohomology complex of A is equal to the direct sum

$$C^*(A) = \bigoplus_{n\geq 0} Hom(A^{\otimes n}, A),$$

with its natural grading and with the differential given by

$$\delta(\boldsymbol{\psi})(a_1, a_2, \dots, a_{p+1}) = a_1\psi(a_2, \dots, a_{p+1}) + \sum_{k=1}^{p} \psi(a_1, \dots, a_k a_{k+1}, \dots, a_{p+1}) \\ + (-1)^{p+1}\psi(a_1, a_2, \dots, a_p)a_{p+1}.$$

It is easy to see that the square of this map is equal to 0; the corresponding cohomology is called *Hochschild cohomology of* A. This object has numerous applications in various branches of Mathematics; one of the reasons of this is that the complex (and cohomology) has various algebraic atructures. The structure, which we shall now be interested in is the following: for $\varphi \in C^p(A)$, $\psi \in C^q(A)$ and $1 \leq i \leq p$ put

$$\varphi \circ_i \psi(a_1, \dots, a_{p+q-1}) = \varphi(a_1, \dots, a_{i-1}, \psi(a_i, \dots, a_{i+q-1}), a_{i+q}, \dots, a_{p+q-1}).$$

Then one can define the *Gerstenhaber bracket* $[\varphi, \psi]$ by the formula:

$$[\varphi, \psi] = \sum_{i=1}^{p}(-1)^{i(q-1)}\varphi \circ_i \psi - (-1)^{(p-1)(q-1)}\sum_{j=1}^{q}(-1)^{j(p-1)}\psi \circ_j \varphi.$$

One can show that this map turns the Hochschild complex into a differential graded Lie algebra (with shifted grading, given by $|\varphi| = p - 1$ for $\varphi \in C^p(A)$; this is, certainly, similar to the Lie algebra structure, introduced in the Sect. 2.2, however, these two structures are different, since Gerstenhaber bracket is graded, and the other one is not). We shall also make use of another algebraic structure on the Hochschild complex: for any $\varphi, \psi \in C^*(A)$ one can define their $\cup$-product $\varphi \cup \psi$ by the formula

$$\varphi \cup \psi(a_1, \dots, a_{p+q}) = \varphi(a_1, \dots, a_p)\psi(a_{p+1}, \dots, a_{p+q}).$$

This product turns $C^*(A)$ into a graded differential algebra.

The main motivation for the introduction of Gerstenhaber bracket in the context of deformation theory is the following remark: *Consider a deformation* $*$ *of the product in an algebra* A *given by a formal power series* (*see formula* (1)). *Let* $B = \sum_n \hbar^n B_n$*; we can regard this sum as an element in the space of formal series in* $\hbar$ *with coefficients in* $C^2(A)$. *Then the product is associative iff the following equality holds*

$$\delta B + \frac{1}{2}[B, B] = 0.$$

Here we extend the homological operations to $C^*(A)[[\hbar]]$ linearly in $\hbar$. This equation is called *Maurer-Cartan equation* and the element B, verifying it is called *Maurer-Cartan element.*

This construction can be applied to any associative algebra A, but we shall be interested in the case $A = C^\infty(M)$. This algebra being infinite-dimensional and topological, its Hochschild cohomology as stated above, cannot be computed in a general case. Therefore one usually considers a somewhat restricted version of this complex: put

$$C^*_{loc}(C^\infty(M)) = \bigoplus_n P^n(M),$$

where $P^n(M)$ is the space of polydifferential operators of n variables on M, i.e. $D \in P^n(M)$ is a linear map $D : C^\infty(M)^{\otimes n} \to C^\infty(M)$, which is a differential operator in every argument. Below we shall deal exclusively with this sort of Hochschild complex, so we shall usually omit the subscript *loc*.

In this case one can effectively compute the cohomology (see for instance [11]):

Theorem 4 (Hochschild-Kostant-Rosenberg theorem). *(Local) Hochschild cohomology of the smooth functions on a manifold is isomorphic to the space of polyvector fields* $D^*(M)$ *(sections of the exterior powers of tangent bundle); this isomorphism is induced by a linear map* $\chi_M : D^*(M) \to P^*(M)$*, called the* ***Hochschild-Kostant-Rosenberg map****. Moreover, the Lie algebra structure on cohomology (induced from Gerstenhaber bracket) coincides with the structure, given by Schouten bracket. Similarly, the cup-product on Hochschild complex induces the usual wedge-product of polyvector fields.*

Recall that Schouten bracket is the unique bilinear and graded skew symmetric map on polyvector fields, which verifies the Leibniz rule with respect to the exterior product and coincides with the commutator on vector fields.

Thus we have two differential Lie algebras, $\mathscr{P}^*(M) = (C^*(C^\infty(M)), \delta, [,])$ and $\mathscr{D}^*(M) = (D^*(M), 0, [,])$ (here $[,]$ denotes Gerstenhaber and Schouten bracket, respectively). The main result of Kontsevich can be formulated, loosely speaking, as follows: *there exists a map from* $\mathscr{D}^*(M)$ *to* $\mathscr{P}^*(M)$*, which is almost homomorphism.* It should be remarked here that the Hochschild-Kostant-Rosenberg map χ_M does not commute with the brackets, but Kontsevich shows that there exists a consistent way to choose all higher homotopies, connecting this map with a genuine homomorphism of Lie algebras.

More accurately, one says that there is L_∞-map between two DG Lie algbras, $U : \mathscr{P} \to \mathscr{Q}$, if there is a collection of linear maps $U_n : \Lambda^n \mathscr{P} \to \mathscr{Q}$ of degree $1-n$ each, verifying the following set of relations

$$\begin{aligned}&\partial U_{n+1}(a_1, \dots, a_{n+1})\\ &\quad = \sum_{1\le i<j\le n+1} (-1)^{\epsilon(i,j)} U_n([a_i, a_j], a_1, \dots, \widehat{a_i}, \dots, \widehat{a_j}, \dots, a_{n+1})\\ &\qquad + \sum_{p+q=n+1} \sum_{\sigma\in Sh(p,q)} (-1)^\sigma [U_p(a_{\sigma(1)}, \dots, a_{\sigma(p)}), U_q(a_{\sigma(p+1)}, \dots, a_{\sigma(n+1)})].\end{aligned}$$

In particular, every L_∞-map induces a homomorphism of cohomology Lie algebras. One can show, that the L_∞-map, which induces an isomorphism in cohomology is always invertible (up to homotopy) as L_∞-map. One can also show, that every Maurer-Cartan element π in $\mathscr{P}$ induces with the help of an L_∞-map U a Maurer-Cartan element $U(\pi)$ in $\mathscr{Q}$, provided the folowing series is convergent

$$U(\pi) = \sum_{n=1}^\infty \frac{1}{n!} U_n(\pi, \dots, \pi).$$

Kontsevich gives an explicit formula for the higher terms U_n for such a map, so that one can take $U_1 = \chi_M$; in the case we consider the convergence is provided by the growing degree of $\hbar$ in the formula.

In addition to the role this quasi-isomorphism plays in the construction of the star-product, it can be used to transfer certain elements from classical to quantized algebra. For instance, it is not difficult to see that the formula

$$\hat{f} = \sum_{n=0}^\infty \frac{\hbar^n}{n!} U_{n+1}(f, \pi, \dots, \pi)$$

for any $f \in Z_\pi(C^\infty(X))$ determines an element in the center of $\mathscr{A}(X)$. This approach was used, for instance, by Calaque and van den Bergh in the paper [12], in their proof of Duflo isomorphism and its generalizations.

3.2 Garay-Van Straten Cohomology and Obstruction Theory

One of the first attempts to apply the cohomological methods to the question of deformation quantization of commutative subalgebras is in the paper [13]. Let us briefly recall the main constructions from this paper.

Let $f_1, \dots, f_n$ be a finite family of commuting (complex-valued) functions on a Poisson manifold; here commutativity is understood in the sense of Poisson bracket. We try to find the corresponding commutative (in the usual sense) family of elements $F_1, \dots, F_n \in C^\infty(M)[[\hbar]]$, so that $F_i \equiv f_i \mod (\hbar)$; More accurately, we assume that F_i has the following form

$$F_i = f_i + \sum_p \hbar^p f_{i,p},$$

where $f_{i,p} \in C^\infty(M)$, $i = 1, \dots, n$, $p \in \mathbb{N}$ are chosen so that the equations $[F_i, F_j] = 0$ hold for all i and j.

We shall construct the elements $f_{i,p}$ by induction in p, starting with $f_{i.0} = f_i$. Let's begin with the first step:

$$[f_i + \hbar f_{i,1}, f_j + \hbar f_{j,1}] = \hbar\{f_i, f_j\} + \hbar^2 \left(\bar{B}_2(f_i, f_j) + \{f_i, f_{j,1}\} + \{f_{i,1}, f_j\}\right) + \dots$$

(we omit the commutators with respect to the commutative multiplication, which is clearly equal to 0); here $\bar{B}_2$ denotes the antisymmetrization of the map B_2 in the deformation series and the dots in the end stand for the terms of degree 3 and higher in $\hbar$. Since f_i, f_j were assumed to be commutative, we come up with a system of PDE, which should hold, if the commutators of F_i and F_j vanish in degrees 1 and 2 (in $\hbar$):

$$\{f_i, f_{j,1}\} + \{f_{i,1}, f_j\} = -\bar{B}_2(f_i, f_j). \tag{4}$$

Similarly, if we assume, that we have found $f_{i,1}$, $i = 1, \dots, n$, so that the equations above hold, we can write down the equations for the next step:

$$\begin{aligned}[][f_i + \hbar f_{i,1} + \hbar^2 f_{i,2}, f_j + \hbar f_{j,1} + \hbar^2 f_{j,2}] = \hbar^3 \Big(&\bar{B}_3(f_i, f_j) + \bar{B}_2(f_i, f_{j,1}) \\ &+ \bar{B}_2(f_{i,1}, f_j) + \{f_{i,1}, f_{j,1}\} + \{f_i, f_{j,2}\} + \{f_{i,2}, f_j\}\Big) + \dots\end{aligned}$$

Here we have used the fact, that all the terms with lower degrees cancel by our assumption (once again, dots in the end denote the sum of terms with higher degrees). Thus, the commutator vanishes in degree 3 iff the following equation holds:

$$\{f_i, f_{j,2}\} + \{f_{i,2}, f_j\} = -(\bar{B}_3(f_i, f_j) + \bar{B}_2(f_i, f_{j,1}) + \bar{B}_2(f_{i,1}, f_j)). \tag{5}$$

One can continue in this manner, obtaining an infinite series of systems of PDEs, so that $f_{i,n}$ will be solutions to these systems. In the paper [13] these equations were described in cohomological terms as follows.

Consider the following complex, which is called "Koszul complex" in the cited paper and denoted $C_f^\cdot$: let $f_1, f_2, \dots, f_n$ be the Poisson-commuting system of functions in $C^\infty(M)$, we put

$$C_f^\cdot = C^\infty(M) \otimes \Lambda^\cdot(\mathbb{C}^n)$$

as (graded) linear spaces. Let $e_1, \dots, e_n$ be the canonical basis in $\mathbb{C}^n$, then the differential in $C_f^\cdot$ is given by the formula

$$\delta(m \otimes \omega) = \sum_{i=1}^n \{f_i, m\} \otimes e_i \wedge \omega.$$

Here $m \in C^\infty(M)$ and $\omega \in \Lambda^\cdot(\mathbb{C}^n)$. It is easy to see that $\delta^2 = 0$ (it follows from the Jacobi identity). Now, let the right hand sides of Eqs. (4) and (5) etc.

be denoted by χ_{ij}^k (here i, j denote the indices of the functions involved, and k is the number of the step). Then one can define the *k-th anomaly class* of the problem as the class in the second cohomology of $C_f^{\cdot}$, determined by the cocycle

$$\chi_k = \sum_{i,j} \chi_{ij}^k \otimes e_i \wedge e_j$$

(the fact that it is closed also follows from the Jacobi identity). The following theorem is proved in the cited paper.

Theorem 5. *Let $F_i^{(k)} = f_i + \hbar f_{i,1} + \dots + \hbar^k f_{i,k}$ be some extensions of f_i ($i = 1, \dots, n$), for which the equalities $[F_i^{(k)}, F_j^{(k)}] = 0$ hold up to the degree k. Then one can extend them to extensions $F_i^{(k+1)}$ so that $[F_i^{(k+1)}, F_j^{(k+1)}] = 0$ up to $\hbar^{k+1}$ iff the corresponding anomaly class $[\chi_k] = 0$ in $H^2(C_f^{\cdot}, \delta)$.*

In general, however, these cohomology groups (and in particular, classes $[\chi_k]$) are hard to compute. Moreover, the computation depend on the choice of the generators $f_1, \dots, f_n$ in a Poisson-commutative algebra.

Another important result of the paper [13] is the following criterion, providing a simpler way to resolve the question of quantization in a particular case when $X = \mathbb{R}^{2n}$ and we restrict our attention to the polynomial complex-valued functions on it, $\pi = \sum_{k=1}^n \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial q_k}$ and the commutative family consists of n polynomials with an additional condition, that $\mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$ is flat module over the free commutative algebra, generated by $f_1, \dots, f_n$.

In order to explain this criterion consider the action of the polynomial algebra $\mathbb{C}[t_1, \dots, t_n]$ on the complex $C_f^{\cdot}$, given by the formula $t_i(m \otimes \omega) = (f_i m) \otimes \omega$. Clearly, this action commutes with the differential, so we obtain the structure of a (graded) $\mathbb{C}[t_1, \dots, t_n]$-module on the cohomology $H^*(C_f^{\cdot}, \delta)$. Then the following statement is the main result of [13].

Theorem 6. *If the module $H^2(C_f^{\cdot}, \delta)$ is torsion-free, then all the anomaly classes vanish and there exists a deformation quantization of the integrable system $f_1, \dots, f_n$.*

3.3 Relative Hochschild Cohomology and Relative Poisson Cohomology

A construction, slightly more invariant than that of Garay and van Straten was given in the paper [14]. Let us briefly explain it here.

It begins with the observation, that instead of looking for the continuation of functions in $C^\infty(M)$, which commute with respect to the Poisson bracket, one can look for such deformation series $\sum_{n=1}^\infty \hbar^n B_n(f, g)$ that will vanish identically for all f, g in the commutative subalgebra. Since all the star-products, corresponding to the same Poisson brackets are equivalent, we can now look for the corresponding "gauge transformation" $T(f) = f + \hbar T_1(f) + \hbar^2 T_2(f) + \dots,$

where T_k are suitable differential operators on M such that the corresponding star-product

$$f *' g = T^{-1}(T(f) * T(g)) \tag{6}$$

will coincide with the usual commutative product on the subalgebra $C \subseteq C^\infty(M)$, i.e. such that

$$f * g = fg, \ \forall f, g \in C. \tag{7}$$

Once again this is done by an inductive process beginning with T_1: we rewrite the equalities (6) and (7) as $T(fg) = T(f) * T(g)$ for all $f, g \in C$, then in degree 1 we have

$$fT_1(g) - T_1(fg) + T_1(f)g = 0$$

that is $\delta T_1(f, g) = 0$ for all $f.g \in C$ (the right hand side vanishes identically since $\{f, g\} = 0$ for $f, g \in C$). Observe, that this equation can hold only when the arguments of δT_1 are taken from C and not always. At the next step we get (for all functions from C)

$$\delta T_2 = T_1 \cup T_1 + [B_1, T_1],$$

where on the right hand side we use the notation from previous section. We can continue in this manner: assume that $T_1, \ldots, T_{n-1}$ has been chosen so that the corresponding $*'$-product $f *' g$ vanishes up to degree $n-1$ in $\hbar$ when f and g are from C. We may now fix the corresponding lower-degree part of the transformed star-product $*'$, i.e. $B'_1, \ldots, B'_{n-1}$ such that $B'_k(f, g) = 0$ for all $f, g \in C$ and look for T_n, such that the "gauge transformation" $T(f) = f + \hbar^n T_n(f) + o(\hbar^{n+1})$ will take $*'$ to a product $*''$ in which $B''_n(f, g) = 0$ for all $f, g \in C$. This will give us the following condition:

$$\delta T_n = B'_n + [B_1, T_{n-1}]. \tag{8}$$

Let us denote the right hand side of this equality by ω_n; then the condition we have means, that this Hochschild cochain in $C^2(C, C^\infty(M)) \subseteq C^2(C^\infty(M))$ is equal to 0 (here $C^*(C, C^\infty(M))$ denotes the Hochschild complex of C with coefficients in $C^\infty(M)$ as in C-bimodule; the definition is identical to the one, given in Sect. 3.1, except that now we allow the maps take values in a larger algebra $C^\infty(M)$ rather than in C). This brings us to the following conclusion.

Proposition 1. *There exists a "gauge transformation" $T(f)$, such that the new product will coincide with the usual one on C iff all the classes $[\omega_n] \in H^2(C, C^\infty(M))$ vanish.*

This construction is rather cumbersome, since in general the relative Hochschild cohomolgy is rather big and difficult to compute. A particular case, when these cohomology groups can be found in more or less explicit terms is given by the following construction.

Let X be a smooth manifold, endowed with trivial Poisson structure. Let $p : M \to X$ be a map of Poisson manifolds, i.e. which intertwines the Poisson structures. Put $C = p^*(C^\infty(C)) \subseteq C^\infty(M)$; then C is a commutative Poisson subalgebra in $C^\infty(M)$. Also assume, that p is a locally trivial fibre bundle. Than we can consider the subbundle $T^{vert}(M_p) \subset TM$, consisting of those vectors in TM, which project to 0 by p. Taking quotient of TM by this subbundle, we obtain the "horizontal" bundle of p, denoted by $T^{hor}M_p$. Then we have the following theorem, which one can call "the relative Hochschild Kostant Rosenberg theorem".

Theorem 7. *The relative Hochschild cohomolgy $H^*(C, C^\infty(M))$ of the algebra $C = p^*(C^\infty(X))$ with coefficients in C-bimodule $C^\infty(M)$ is equal to the algebra of horizontal polyvector fields on M, i.e. to the algebra $\Lambda^*(T^{hor}M_p)$ of sections of the exterior powers of $T^{hor}M_p$.*

With this theorem in mind we can make the following observation, which relates the obstructions from this section to Garay and van Straten's results. Namely, suppose that $X = \mathbb{R}^n$ (in effect, this is always so at least locally, so this construction is applicable in a wider range of cases). Then one can perform explicit computations of the relative Hochschild cohomology, using the Koszul complex as a free C-bimodule resolution of $C^\infty(M)$ (or use the results of the relative HKR theorem and choose a cross-section of p to trivialize the bundle $T^{hor}M_p$) and see that $H^*(C, C^\infty(M))$ is isomorphic (as graded space) to the Garay and van Straten complex.

On the other hand, one can see that Gerstenhaber bracket with the Poisson bivector π induces a chain map on the Hochschild complex $C^*(C, C^\infty(M))$ and so it induces a map in cohomology, given by Schouten bracket of a horizontal polyvector field with Poisson bivector: it is easy to see that this bracket sends vertical fields to vertical, so it induces a map on $\Lambda^*(T^{hor}M_p)$. Since $[\pi, \pi] = 0$, square of this map is equal to 0. In general, recall that Schouten bracket with a Poisson bivector induces a differential on the space of polyvector fields on a Poisson manifold M, called the Lichnerowicz-Poisson differential; the corresponding cohomology is called "Poisson cohomology" of M. If the Poisson structure is non-degenerate (i.e. if M is in effect a symplectic manifold), this cohomology is isomorphic to the usual de Rham cohomology of M. In our case we obtain a "relative" variant of this cohomology, which we shall denote $HP^*(M, p)$. It is easy to show, that the relative Poisson differential is equal to the Garay and van Straten differential under the identification we described above, so Garay and van Straten cohomology is equal to the (particular case of) relative Poisson cohomology.

Finally, let us observe, that Eq. (8) determine T_n up to an addition of a Hochschild cocycle, so that we can modify T_n, if necessary, on the next step. Then the equation on the next step can in turn be reinterpreted as the condition that B_n is equal to Poisson coboundary of T_{n-1} (up to the Hochschild coboundary). Thus we obtain the following result

Proposition 2. *There exists a deformation quantization, which preserves the commutativity of C if the classes $[B_n]$ (iteratively constructed) in relative Poisson cohomology are trivial.*

Clearly, in case $X = \mathbb{R}^n$, this is precisely the condition of Garay and van Straten.

3.4 Vector Fields and Lie Algebra Cohomology

One can apply the ideas and methods developed in previous sections to the following problem (a generalization of the question we study in this survey): *is it possible to raise an action of Lie algebra from $C^\infty(M)$ to its deformation* $\mathscr{A}(M) = (C^\infty(M)[[\hbar]], *)$. This has been done in the paper [15], here we give the outline of its results.

More accurately, let $\mathfrak{g}$ be a finite-dimensional Lie algebra, acting by vector fields on M, i.e. there is a map $\rho : \mathfrak{g} \to Vect(M)$, verifying the condition

$$[\rho(X), \rho(Y)] = \rho([X, Y]).$$

We are looking for a map $\hat{\rho} : \mathfrak{g} \to Der(\mathscr{A}(M))$, such that

$$\hat{\rho}(X) = \rho(X) + \hbar\rho_1(X) + ...$$

Here as before dots in the end denote the terms with $\hbar^k$, $k \geq 2$. We also ask for the condition

$$[\hat{\rho}(X), \hat{\rho}(Y)] = \hat{\rho}([X, Y]).$$

One can try to solve this problem as before by iterations.

For instance, if the algebra is 1-dimensional, we just need to find an extension

$$\hat{\xi} = \xi + \hbar\xi_1 + \dots$$

of a vector field ξ on M to a differentiation $\hat{\xi}$ of $\mathscr{A}(M)$. The first step of the iterative process gives us the relation

$$\delta(\xi_1) = [B_1, \xi].$$

Now, on the right hand side we have the Hochschild 2-cochain, determined by the bivector field, equal to the Schouten bracket $[\pi, \xi]$, i.e. this map is skew-symmetric with respect to the arguments, while on the left we have a symmetric map (since the algebra $C^\infty(M)$ is commutative). Thus we conclude that both sides are trivial and in particular ξ is a Poisson field, i.e. such a field that $L_\xi\pi = 0$.

Further analysis shows that the generic equation at the n-th step has form

$$\delta\xi_n = \sum_{p+q=n} [B_p, \xi_q]. \tag{9}$$

Thus the necessary and sufficient condition for the existence of ξ_n is that the right hand side of this formula determines a trivial class in the second Hochschild cohomology.

Besides this, similarly to the discussion of the previous section, we observe, that the first term in the expression on the right hand side of Eq. (9), which is equal to the Gerstenhaber bracket $[B_1, \xi_{n-1}]$, will change by an element, equal to the Poisson differential, if we change ξ_{n-1}, so that the similar equality on the previous step is not broken. Thus, we come up with the following statement.

Proposition 3. *There exists a derivative $\hat{\xi} \in Der(\mathscr{A}(M))$, extending the given Poisson vector field ξ, if the cocycles $\sum_{p+q=n}[B_p, \xi_q]$ determine trivial elements in the Poisson cohomology of M.*

One should be a little careful here: in effect, the question, answered by this construction, is whether it is possible to extend the given Poisson field all the way up to a derivation by (probably) changing at every step only the last chosen extension. Of course, there could exist much more intricate ways to go from a field to the derivation. In paticular, it is easy to see that if we use Kontsevich's formality map to define the star-product, then we can use it to determine such extension for any ξ: just put

$$\hat{\xi} = \sum_{n \geq 0} \frac{\hbar^n}{n!} U_{n+1}(\xi, \pi, \ldots, \pi),$$

where on the right hand side we have the bivector π repeated n times (compare this formula with the formula for central elements in the end of Sect. 3.1). Then it is easy to see, using the definition of the L_∞-map that $\hat{\xi}$ is a differentiation of the $*$-product.

However, this method fails, if we apply it to a Lie algebra $\mathfrak{g}$ of dimension greater than 1 (acting on M by Poisson vector fields). It is easy to see that if we let $\hat{\rho}$ be given by

$$\hat{\rho}(\xi) = \sum_{n \geq 0} \frac{\hbar^n}{n!} U_{n+1}(\rho(\xi), \pi, \ldots, \pi),$$

then we shall have

$$[\hat{\rho}(\xi), \hat{\rho}(\eta)] - \hat{\rho}([\xi, \eta]) = Ad\left(\sum_{n \geq 0} \frac{\hbar^n}{n!} U_{n+2}(\rho(\xi), \rho(\eta), \pi, \ldots, \pi)\right).$$

If we denote the composition of the map on the right with the natural projection onto the commutatnt space $\mathscr{T}(M) = \mathscr{A}(M)/[\mathscr{A}(M), \mathscr{A}(M)]$ (the universal trace on $\mathscr{A}(M)$) by $\Phi(\xi, \eta)$, then it will give us a 2-cocycle on $\mathfrak{g}$ with values in $\mathscr{T}(M)$. The corresponding cohomology class $[\Phi] \in H^2(\mathfrak{g}, \mathscr{T}(M))$ vanish, if there exists a homotopic L_∞-map U', for which the right hand side of the last formula will vanish.

Alternatively, one can try to construct the map $\hat{\rho}$ again by iterations. This will give us a series of elements in the 2-dimensional cohomology of $\mathfrak{g}$ with values in the Poisson cohomology complex of the manifold. More accurately, these classes lie in the cohomology of truncated double complex $\tilde{C}^*(\mathfrak{g}, CP^*(M))$ given by

$$\tilde{C}^*(\mathfrak{g}, CP^*(M)) = \bigoplus_{p+q=n, p,q>0} C^p(\mathfrak{g}, CP^q(M)),$$

where $CP^*(M)$ is the Lichnerowicz Poisson cohomology complex of M. These classes should vanish, if there exists an extension of ρ to a representation on the quantized algebra. Some details can be found in [15].

3.5 Examples

Let us give few examples of integrable systems and their quantization. We are going to use the methods and ideas from the previous sections.

Our main interest is applications to the universal enveloping algebra. Namely, let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual space. Clearly the algebra $S(\mathfrak{g})$ can be regarded as the algebra of (polynomial) functions on $\mathfrak{g}^*$; then the Lie bracket on $\mathfrak{g}$ can be interpreted as the Poisson structure on this algebra. It has been shown (see e.g. [3]) that the deformation quantization of $S(\mathfrak{g}^*)$ is closely related to the universal enveloping algebra of $\mathfrak{g}$. Thus, if the deformation problem for a commutative system in $S(\mathfrak{g}^*)$ can be solved, in the end we shall find a commutative subalgebra inside $U(\mathfrak{g})$.

Example 5. Let $\mathfrak{g} = \mathfrak{so}(3)$, i.e. this is a 3-dimensional algebra with basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and relations

$$[\mathbf{e}_i, \mathbf{e}_j] = \epsilon_{ijk}\mathbf{e}_k.$$

We shall denote the generators of $S(\mathfrak{g})$, corresponding to the elements $\mathbf{e}_i$ by x, y and z, then the Poisson bracket in $S(\mathfrak{g}) = \mathbb{R}[x, y, z]$ is given by the relations

$$\{x, y\} = z,\ \{y, z\} = x,\ \{z, x\} = y,$$

or

$$\pi = x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x + z\partial_x \wedge \partial_y,$$

where $\partial_x, \partial_y, \partial_z$ are the partial derivatives with respect to x, y and z (below we shall denote $\partial_x = \partial_1$ etc.).

Let us consider the following pair of Poisson commuting elements in this algebra: $f = x^2 + y^2,\ g = z$; indeed

$$\{f, g\} = 2x\{x, z\} + 2y\{y, z\} = -2xy + 2xy = 0.$$

In order to quantize this pair let us look at the first obstruction of Garay and van Straten: we are looking for f_1 and g_1 so that

$$\{f, g_1\} + \{f_1, g\} = -\bar{B}_2(f, g).$$

An easy computation, using the formula:

$$B_2(f,g) = \frac{1}{4}\pi^{ij}\pi^{kl}\partial_i\partial_k(f)\partial_j\partial_l(g) + \frac{1}{6}\pi^{ij}\partial_j\pi^{kl}(\partial_i\partial_k(f)\partial_l(g) - \partial_k(f)\partial_i\partial_l(g)),$$

shows that the first obstruction vanishes identicaly on $\mathfrak{g}$ (in effect, this bidifferential operator is symmetric, so its' antisymmetrization vanishes identically; also observe that this expression differs a little bit from Kontsevich's formula, but one can easily show that the cohomology class does not change, since all star product associated with the same Poisson structure are equivalent). Thus, we can take $f_1 = g_1 = 0$ in our deformation.

Further, a straightforward analysis of Kontsevich's formula shows, that in the case of linear bivector π (which is true in the case of Lie algebra) all the higher degree elements $B_k,\ k \geq 3$ are differential operators of degree at least 4 (the only graph, corresponding to the operator of degree 3 in the expression for B_3, comes with trivial coefficient), so $\bar{B}_k(f,g) = 0$ for all k. So we can take all $f_k = g_k = 0$ for all k. This means, that

$$f * g = g * f$$

for Kontsevich's quantization, and no correction terms are necessary in this case. Also observe, that this reasoning goes word by word for any Lie algebra and any pair of commuting functions that are polynomials of degrees 2 and 1 on it.

It is instructive however to take a look at the Garay and van Straten cohomology in this case. Namely, we are interested in the second cohomology group $H^2(C_f, \delta)$. Since there are only two commuting functions, $\delta = 0$ on the second degree part of the complex. On the other hand,

$$\delta(a \otimes e_1 + b \otimes e_2) = (\{g,b\} - \{f,a\}) \otimes e_1 \wedge e_2.$$

An easy computation with bivector π shows that

$$\{g,b\} = (x\partial_y - y\partial_x)(b)$$

and

$$\{f,a\} = 2z(x\partial_y - y\partial_x)(a).$$

If we introduce the cylindrical coordinates r, φ, z on $\mathbb{R}^3$ so that z coincides with the usual z coordinate, $x = r\cos\varphi$, $y = r\sin\varphi$ then we have

$$\delta(a \otimes e_1 + b \otimes e_2) = (\partial_\varphi b - 2z\partial_\varphi a) \otimes e_1 \wedge e_2.$$

Thus, $\mathrm{Im}(\delta) = \mathrm{Im}(\partial_\varphi)$. If we consider the Fourier decomposition with respect to φ (and functions, which have such decomposition are dense in $C^\infty(\mathbb{R}^3)$) we shall see that

$$H^2(C_2, \delta) \cong C^\infty(\mathbb{R}^3)/\mathrm{Im}(\partial_\varphi) = C^\infty(\mathbb{R}^3)^{S^1},$$

where on the right we have the space of all functions, invariant with respect to the rotations of $\mathbb{R}^3$ around the Oz axis.

Finally, we observe, that the $\mathbb{C}[t_1, t_2]$ module structure on $C^\infty(\mathbb{R}^3)^{S^1}$ is induced by multiplications by $f = r^2$ and z, which are clearly functionally independent (in fact, polynomials in r and z are dense in $C^\infty(\mathbb{R}^3)^{S^1}$). Thus the cohomology $H^2(C_f, \delta)$ is torsion free and the Theorem 6 is applicable although, of course, our case is not of the form, prescribed by this theorem, e.g. the space we consider is odd-dimensional. It is an interesting question, what is the possible scope of applicability for this theorem.

Let us again mention, that the reasoning we gave here can be applied to arbitrary Lie algebra, which means that any family of commutative elements, which are of degree 1 and 2 on a Lie algebra (so that there is not more than one element of degree 2) is quantizable.

Example 6. Let us now consider $M = \mathbb{R}^{2n}$ with constant Poisson structure, e.g. $\pi = \sum_{k=1}^{n} \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial q_k}$, where $(p_1, \dots, p_n, q_1, \dots, q_n)$ are coordinates in $\mathbb{R}^{2n}$. In this case one can show, that any family of commuting polynomials of degree 2 is quantizable "on the nose", i.e. that if $\{f, g\} = 0$ for two such polynomials, than $f * g = g * f$.

To see this, we consider the "anomalies" of Garay and van Straten again: the first anomaly, equal to the antisymmetrization of $B_2(f, g)$ vanishes, because the operator

$$B_2 = \frac{1}{4} \pi^{ij} \pi^{kl} (\partial_i \partial_k \otimes \partial_j \partial_l)$$

is symmetric again. Further, the remainig obstruction classes will be given by application of the operators

$$B_k = \frac{1}{2^k} \sum_{i_1, \dots, i_k, j_1, \dots, j_k} \pi^{i_1 j_1} \dots \pi^{i_k j_k} (\partial_{i_1} \dots \partial_{i_k} \otimes \partial_{j_1} \dots \partial_{j_k}),$$

which have degrees greater, than 2 in both arguments, thus they will also vanish identically.

Example 7. We finally consider the situation, in which the commuting pair of elements consists of two functions f and g, and we assume, that one of them, for instance f, verifies the condition $\{f, C^\infty(M)\} = C^\infty(M)$. It turns out that in this case, we can again find the quantization $\hat{f}$ and $\hat{g}$ so that $\hat{f} * \hat{g} = \hat{g} * \hat{f}$.

In fact, one can take $f_1 = f_2 = \dots = f_k = 0$. Indeed, assume, that we have chosen $g_1, \dots, g_k$ so that $f * \hat{g}_k = \hat{g}_k * f$ (here $\hat{g}_k = g + \hbar g_1 + \dots + \hbar^k g_k$), then the equation, that determines g_{k+1} will be

$$\{f, g_{k+1}\} = -\bar{B}_{k+1}(f, g).$$

It follows from the assumption we made, that this equation always has solution, so we can continue reasoning by induction.

A particular case of this situation is when the Hamiltonian vector field, generated by f can be integrated to a global coordinate on M (for instance, if $M = \mathbb{R}^n$). Then one can always solve the equation $\{f, h\} = g$ for any right hand side g, simply by integrating along this coordinate. For example, if the field $\{f, \cdot\}$ is nowhere vanishing, this can always be done locally. It is interesting, whether (and under what conditions) this construction can be globalized.

4 Conclusion: Remarks and Questions

We are going to conclude this paper with few questions and remarks, which might be helpful for future work, and we hope that some of the readers will be interested enough to solve some of them.

First of all, it seems to be extremely instructive to compute more good examples of commutative algebras and obstructions for their quantization. So far, in the examples discussed in previous section, we only dealt with low-degree polynomial subalgebras in $S(\mathfrak{g})$ and $\mathbb{C}[x_1,\ldots,x_n]$. It is not very surprising that all these examples turned out to be easily quantizable. It would be very interesting to find a counterexample to quantizability on these grounds.

Next, let $C \subseteq C^\infty(X)$ be a Poisson commutative subalgebra; consider the exact sequence of local Hochschild complexes,

$$0 \to IC^*(C, C^\infty(X)) \to C^*(C^\infty(X)) \to C^*(C, C^\infty(X)) \to 0$$

induced by the inclusion of C. Here, $IC^*(C, C^\infty(X))$ is the kernel, and we assume that any $C^\infty(X)$-valued polydifferential operator on C is a restriction of an operator on $C^\infty(X)$ (which is the case, for instance, if C is a pull-back algebra of functions from the base of a locally-trivial bundle). This construction is closely related with the construction of Sect. 3.3. Then, the question is, if the properties of this exact sequence can be related to the obstructions, we construct in this section? In particular, how is its extension class related to the question we pose?

Next, as we observed in the previous section, in certain circumstances the quantization problem can be solved locally. For example, if the Hamiltonian vector field $X_f = \{f, -\}$ is nonzero at some point $x_0 \in X$, than for any function g, such that $\{f, g\} = 0$, the quantization problem can be solved in a vicinity of x_0: under the aforesaid condition, we can find a local coordinate t near x_0 so that the fields $\frac{\partial}{\partial t}$ and X_f would coincide, then we see that the condition $\{f, C^\infty(U)\} = C^\infty(U)$ holds for a suitable open neighbourhood of x_0. Thus, the construction of the last example from the Sect. 3.5 is applicable in U. This gives us a pair of commuting elements $\hat{f}_U$, $\hat{g}_U$ in $\mathscr{A}(U)$. On the other hand, since the star-product is given by local formulas, the obstruction classes, constructed in previous sections, can be restricted to U. Thus we have to check the relation of local classes (which vanish) and the restrictions of the classes, defined on all X: the question is, if one can derive global quantization from the local one. And if it is true, then it is interesting, how the global elements $\hat{f}$, $\hat{g}$ can be constructed from the local formulas. A special case, when the fields X_f, X_g vanish in isolated points, is of special interest, since it implies that the properties of the elements $\hat{f}$, $\hat{g}$ will depend on the properties of the Liouville tori foliation around these singular points.

Finally, there is a question, relating the argument shift method and the homological constructions from the previous sections. Namely, the relation, which is crucial to the argument shift method,

$$L_\xi^2 \pi = 0$$

seems to have certain homological meaning, which needs to be investigated. One of the possible approaches to it, is to rewrite it as

$$[[\pi, \xi], \xi] = 0,$$

where on the right stand the Schouten brackets.

Then the left hand side of this equality can be interpreted as the derived bracket $[\xi, \xi]_\pi$ in $D^*(X)$

$$[a, b]_\pi = [[\pi, a], b]$$

(see the paper [16], in which the notion of derived brackets in differential Lie algebras is discussed at leisure). This bracket verifies all the properties of the graded Lie bracket, except for the graded skew-symmetry. Algebras with such brackets are called *Leibniz algebras*. Thus, one of the possible approaches to the deformation quantization of the argument shift method is to investigate the relation of the Leibniz algebra $(D^*(X), [,]_\pi)$ and the star-product. An alternative approach would be to use the ungraded Lie algebra structure from Sect. 2.2 and look for the corresponding homotopy algebra properties.

Finally, observe, that there exist many other possible approaches to the deformation quantization of integrable systems. For instance, one can use alternative quantization procedures: for example, there exists Gutt's quantization formula for the quantization of Lie algebras (see [17]) and Fedosov's quantization [18], phrased in geometric terms. It would be interesting to see the relation between these results and the integrable systems. One can also look for the deformation constructions within the classical R-matrix formalism, and quantum groups. All these questions seem to be quite interesting and we believe, that a deeper understanding of relations between them can be fruitful in many other branches of Mathematics.

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Graded Thread Modules Over the Positive Part of the Witt (Virasoro) Algebra

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Abstract. We study $\mathbb{Z}$-graded thread W^+-modules $V = \oplus_i V_i$, dim $V_i = 1, -\infty \le k < i < N \le +\infty$, $\dim V_i = 0$, otherwise,over the positive part W^+ of the Witt (Virasoro) algebra W. There is well-known example of infinite-dimensional $(k = -\infty, N = \infty)$ two-parametric family $V_{\lambda,\mu}$ of W^+-modules induced by the twisted W-action on tensor densities $P(x)x^\mu(dx)^{-\lambda}, \mu, \lambda \in \mathbb{K}, P(x) \in \mathbb{K}[t]$. Another family $C_{\alpha,\beta}$ of W^+-modules is defined by the action of two multiplicative generators e_1, e_2 of W^+ as $e_1 f_i = \alpha f_{i+1}$ and $e_2 f_j = \beta f_{j+2}$ for $i, j \in \mathbb{Z}$ and α, β are two arbitrary constants ($e_i f_j = 0, i \ge 3$).
We classify $(n + 1)$-dimensional graded thread W^+-modules of three important types for sufficiently large n. New examples of graded thread W^+-modules different from finite-dimensional quotients of $V_{\lambda,\mu}$ and $C_{\alpha,\beta}$ are found.

Keywords: Witt algebra · Visaroro algebra · Lie algebras

1 Positive Part of the Witt Algebra and its Finite-Dimensional Graded Modules

The representation theory of the Virasoro algebra Vir was intensively studied in the 80s of the last century, now one may consider it as a classical part of the representation theory of infinite-dimensional Lie algebras. For instance, the structure of Verma modules over Virasoro algebra and Fock modules over Vir was completely found out by Mathieu [11] proved V. Kac's conjecture which says that any simple $\mathbb{Z}$-graded Vir-module with finite-dimensional homogeneous components is either a highest weight module, a lowest weight module, or a

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module of the type $V_{\lambda,\mu}$. However, two particular cases of the theorem were already proved: the classification of Harish-Chandra modules for which all the multiplicities of weights are 1 (by Kaplansky and Santharoubane [9]) and the classification of unitarizable Harish-Chandra modules (by Chari and Pressley [2]). Some partial results on Kac's conjecture were obtained in [10].

It should be noted that a great number of well-known mathematicians and mathematical physicists contributed to the development of the theory of representations of the Virasoro algebra, for complete survey we recommend the monograph [8].

In 1992, Benoist, answering negatively the Milnor's question [14] on left-invariant affine structures (flat affine connections) on nilpotent Lie groups, presented examples of compact 11-dimensional nilmanifolds that carry no complete affine structure. For that, he constructed examples of 11-dimensional nilpotent Lie algebras with no faithfull linear representations of dimension 12. In his proof [3] he classified $\mathbb{N}$-graded Lie algebras $\mathfrak{a}_r$ defined by two generators e_1 and e_2 of degrees 1 and 2, respectively, and two relations $[e_2, e_3] = e_5$ and $[e_2, e_5] = re_7$, where r is an arbitrary scalar and $e_{i+1} = [e_1, e_i]$ for all $i \geq 2$.

Lemma 1. (Benoist, [3]) *If $r \neq \frac{9}{10}, 1$, then $\mathfrak{a}_r$ is a finite-dimensional Lie algebra.*

(1) Let $r = \frac{9}{10}$, then $\mathfrak{a}_r \cong W^+$, positive part of the Witt (Virasoro) algebra, it is infinite-dimensional Lie algebra with base $\tilde{e}_i, i \geq 1$ and the relations $[\tilde{e}_i, \tilde{e}_j] = (j-i)\tilde{e}_{i+j}$, where $\tilde{e}_i = e_i/(i-2)!$.

(2) Let $r = 1$, then $\mathfrak{a}_r \cong \mathfrak{m}_2$.

(3) Let $r \neq 0, \frac{9}{10}, 1, 2, 3$, then $\mathfrak{a}_r$ is a 11-dimensional filiform Lie algebra.

We recall that a finite-dimensional nilpotent Lie algebra $\mathfrak{g}$ is called filiform if it has the maximal possible (for its dimension) value of nil-index $s(\mathfrak{g}) = \dim \mathfrak{g} - 1$. In its turn the nil-index $s(\mathfrak{g})$ is the length of the descending central series of $\mathfrak{g}$.

The idea of Benoist's construction was the following. Benoist considered the algebra $\mathfrak{a}_{-2}$ of the family $\mathfrak{a}_r$. It is a 11-dimensional filiform Lie algebra. Itself $\mathfrak{a}_{-2}$ admits a complete affine structure because it is positively graded. Benoist considered its filtered deformation $\mathfrak{a}_{-2,s,t}$ that it is not positively graded and he proved that $\mathfrak{a}_{-2,s,t}$ does not admit any faithfull 12-dimensional representation. That is an obstruction for existence of a complete affine structure on the corresponding nilmanifold. The proof of non-existence of a faithfull 12-representation was based in particular on the classification of graded faithfull $\mathfrak{a}_{-2}$-modules.

Despite the fact that the positive part W^+ of the Witt algebra was not used in his proof, Benoist suggested that special deformations of finite-dimensional factors of W^+ can also be used for counterexamples to Milnor's conjecture in higher dimensions $n > 11$. Hence the classification of graded thread W^+-modules is quite necessary for the possible proof.

The aim of this paper is a classification of $(n + 1)$-dimensional graded thread W^+-modules with additional structure restriction $e_n f_1 \neq 0$ which means that the corresponding representation of the n-dimensional quotient $W^+/\langle e_{n+1}, e_{n+2}, \dots, \rangle$ is faithfull.

Also finite-dimensional graded thread W^+-modules played an essential role in explicite constructions of singular Virasoro vectors in [1] and [13]. Besides these graded thread W^+-modules were used for the construction of trivial Massey products in the cohomology $H^*(W^+, \mathbb{K})$ in [5], answering V. Buchstaber's conjecture, that the cohomology $H^*(W^+, \mathbb{K})$ is generated by non-trivial Massey products of one-dimensional cohomology classes. Finaly V. Buchstaber's conjecture was proved for non-trivial Massey products in [12].

The paper is organized as follows. In the Sect. 1 we recall necessary definitions and facts on positive part of the Witt (Virasoro) algebra and its graded modules. In particular we introduce the important class of $(n+1)$-dimensional graded thread W^+-modules with a property $e_n f_1 \neq 0$ mentioned above. We prove the key Lemma 2 and its Corollary stating that we have only three types of $(n+1)$-dimensional graded thread W^+-modules defined by basis $f_1, \dots, f_{n+1}$:

(a) no zeroes of e_1, i.e. $e_1 f_i \neq 0$, $i = 1, \dots, n$;
(b) one zero of e_1, i.e. $\exists! k$, $1 \le k \le n$, $e_1 f_k{=}0$;
(c) two neighboring zeroes of e_1, i.e. $\exists! k, 1{\le}k{\le}n{-}1, e_1 f_k{=}e_1 f_{k+1}{=}0$.

We classify the modules of the type a) (we call them graded thread W^+-modules of the type $(1, 1, \dots, 1)$) in Sect. 3. W^+-modules with one zero, subcase b), so called modules of the type $(1, \dots, 1, 0, 1, \dots, 1)$ are classified in Sect. 4.

Section 5 is devoted to the most interesting case of graded W^+-modules with two neighboring zeroes of e_1. Modules of this type were applied in [12] for the proof of V. Buchstaber's conjecture on Massey products in Lie algebra cohomology $H^*(W^+, \mathbb{K})$ as we mentioned above.

2 Definitions and Examples

The Witt algebra W can be defined by its infinite basis $e_i = t^{i+1}\frac{d}{dt}, i \in \mathbb{Z}$ and the Lie bracket is given by

$$[e_i, e_j] = (j - i)e_{i+j}, i, j \in \mathbb{Z}.$$

The Virasoro algebra Vir is infinite-dimensional Lie algebra, defined by its basis $\{z, e_i, i \in \mathbb{Z}\}$ and commutation relations:

$$[e_i, z] = 0, \forall i \in \mathbb{Z}, \quad [e_i, e_j] = (j - i)e_{i+j} + \frac{j^3 - j}{12}\delta_{-i,j} z.$$

Vir is a one-dimensional central extension of the Witt algebra W (the one-dimensional center is spaned by z).

There is a subalgebra $W^+ \subset W$ spanned by $e_1, e_2, e_3, \dots$, basis vectors with positive subscripts that we call the *positive part of the Witt algebra*.

Definition 1. ([5]) A W^+-module V is called graded thread W^+-module if there exist two integers $k, N, k < N$, or $k = -\infty, N = +\infty$ and a decompostion $V = \oplus_{-\infty}^{+\infty} V_j, j \in \mathbb{Z}$, such that:

$$e_i V_j \subset V_{i+j}, i \in \mathbb{N}, j \in \mathbb{Z}, \; \dim V_i = \begin{cases} 1, & k < i < N, \\ 0, & \text{otherwise} \end{cases},$$

where $e_1, e_2, \ldots, e_k, \ldots$ is a graded basis of the positive part W^+ of the Witt (Virasoro) algebra.

Remark 1. A graded thread W^+-module $V = \oplus_{-\infty}^{+\infty} V_j$ can be defined by its basis

$$f_i, i \in \mathbb{Z},\, k < i < N, \langle f_i \rangle = V_i,$$

and by the set of its structure constants

$$\alpha_i, \beta_j,\, i, j \in \mathbb{Z},\, k < i < N-2, k < j < N-3,$$

such that

$$e_1 f_i = \alpha_i f_{i+1}, \quad e_2 f_j = \beta_j f_{j+2}.$$

Here we remark that W^+ is generated as a Lie algebra by two elements e_1 and e_2.Certainly the constants α_i, β_j can not be arbitrary. They must satisfy certain algebraic relations that we are going to discuss later.

To begin with, we present two infinite-dimensional examples.

- One can take $\alpha_i = \alpha$, $\beta_j = \beta$. It is a W^+-module $C_{\alpha,\beta}$ where all other basic elements $e_i, i \geq 3$ act trivially.
- We have defined basic vectors e_i of W^+ as differential operators $e_i = x^{i+1}\frac{d}{dx}$ on the real (complex) line. One can consider the space $V_{\lambda,\mu}$ of tensor densities of the form $P(x)x^{\mu}(dx)^{-\lambda}$, where $P(x)$ is some polynomial on x and the parameters λ, μ are arbitrary real (complex) numbers. Operator $\xi = f(x)\frac{d}{dx}$ acts on $F_{\lambda,\mu}$ by means of the Lie derivative L_ξ:

$$L_\xi P(x)x^{\mu}(dx)^{-\lambda} = \left(f(x)(P(x)x^{\mu})' - \lambda P(x)x^{\mu}f'(x)\right)(dx)^{-\lambda}.$$

  Taking the infinite basis $f_j = x^{j+\mu}(dx)^{-\lambda}$ of $F_{\lambda,\mu}$ we have the following W^+-action [6]:

$$e_k f_j = (j + \mu - \lambda(k+1))f_{k+j}.$$

  In other words $V_{\lambda,\mu}$ is a twist of the natural action of W on $\mathbb{C}[t, t^{-1}]$. The defining set for a W^+-module $V_{\lambda,\mu}$ is

$$\alpha_i = i + \mu - 2\lambda,\, \beta_j = j + \mu - 3\lambda.$$

Remark 2. The vector space $V_{\lambda,\mu}$ can be regarded as a W-module over the entire Witt algebra W, or, that is equivalent, as a zero-energy Virasoro representation [7]. As W-module $V_{\lambda,\mu}$ is reducible if $\lambda \in \mathbb{Z}$ and β=0 or $\lambda \in \mathbb{Z}$ and β=1 otherwise it is irreducible [7]. The infinite-dimensional W-modules $V_{\lambda,\mu}$ and $V_{\lambda,\mu+m}, m \in \mathbb{Z}$ are isomorphic.

Having an infinite-dimensional graded thread W^+-module $V = \langle f_i, i \in \mathbb{Z}\rangle$ one can construct a $(N-k-1)$-dimensional W^+-module $V(k,N)$ taking a subquotient of V:

$$V(k,N) = \oplus_{i>k}^{\infty} V_i / \oplus_{j>N-1}^{\infty} V_j, \quad V(k,N) = \langle f_{k+1}, \ldots, f_{N-1}\rangle.$$

From now we deal with a $(n+1)$-dimensional graded thread W^+-module V defined by its basis $f_1, f_2, \dots, f_{n+1}$ and a finite set of constants $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{n-1}\}$.

The dual module V^* of a finite-dimensional graded thread W^+-module $V = \langle f_1, f_2, \dots, f_{n+1}\rangle$ has the structure of a graded thread W^+-module with respect to the basis

$$f_1' = f^{n+1}, f_2' = f^n, \dots, f_n' = f^2, f_{n+1}' = f^1,$$

where $f^1, \dots, f^n, f^{n+1}$ is the dual basis in V^* with respect to the basis $f_1, \dots, f_n, f_{n+1}$ of V, $f^i(f_j) = \delta^i_j$. Let $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{n-1}\}$ be the set of structure constants of the W^+-module V with respect to the basis $f_1, \dots, f_{n+1}$. Then the set

$$\{-\alpha_n, \dots, -\alpha_1, -\beta_{n-1}, \dots, -\beta_1\}$$

defines the structure of the dual W^+-module V^*:

$$\begin{aligned} e_1 f_j' &= e_1 f^{n+2-j} = -\alpha_{n+1-j} f^{n+1-j} = -\alpha_{n+1-j} f_{j+1}', \ j = 1, 2, \dots, n; \\ e_2 f_k' &= e_2 f^{n+2-k} = -\beta_{n-k} f^{n-k} = -\beta_{n-k} f_{k+2}', \ k = 1, 2, \dots, n-1. \end{aligned}$$

We recall that the dual W^+-action on V^* is defined by $(g \cdot f)(x) := -f(gx)$, where $g \in W^+, x \in V, f \in V^*$.

For obvious reasons there is no sense in discussing the irreducibility of graded thread W^+-modules: they are all reducible. Instead of irreducibility, one has to discuss indecomposability. Let the defining sets of structure constants of a W^+-module V be of the following type

$$\{\alpha_1, \dots, \alpha_m, 0, \alpha_{m+1}, \dots, \alpha_n, \beta_1, \dots, \beta_{m-1}, 0, 0, \beta_{m+2}, \dots, \beta_{n-1}\}, \ 0 \le m < n$$

Then, V is a direct sum of two W^+-modules V_1 and V_2

$$V = V_1 \oplus V_2, \ V_1 = \langle f_1, \dots, f_{m+1}\rangle, \ V_2 = \langle f_{m+2}, \dots, f_{n+1}\rangle.$$

where V_1 and V_2 have the following defining sets of structure constants

$$\{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_{m-1}\}, \ \{\alpha_{m+1}, \dots, \alpha_n, \beta_{m+2}, \dots, \beta_{n-1}\},$$

respectively. The converse is also true. One can consider a direct sum $V_1 \oplus V_2$ of two finite-dimensional graded thread W^+-modules V_1 and V_2, may be after possible renumbering of the basis vectors from V_2.

Lemma 2. *Let $V = \langle f_1, f_2, \dots, f_{n+1}\rangle$ be a $(n+1)$-dimensional graded thread W^+-module such that*

$$\exists k, p, \ 1 \le k < k+p \le n+1, \ p \ge 2, \quad e_1 f_k = e_1 f_{k+p} = 0.$$

Then $e_n f_1 = 0$.

Proof. Recall that $e_p f_k \in \langle f_{k+p}\rangle$. Then the equalities $e_1 f_k = e_1 f_{k+p} = 0$ imply that

$$(p-1) e_{p+1} f_k = e_1 e_p f_k - e_p e_1 f_k = 0.$$

On the next step we have

$$pe_{p+2}f_k = (e_1e_{p+1}f_k - e_{p+1}e_1f_k) = 0,\ pe_{p+2}f_{k-1} = (e_1e_{p+1}f_{k-1} - e_{p+1}e_1f_{k-1}) = 0.$$

One can suppose by an inductive assumption that $e_{p+s}f_k = \cdots = e_{p+s}f_{k-s+1} = 0$ for some $s, 1 \le s \le k-1$. Then it follows that

$$(p+s-1)e_{p+s+1}f_k = 0,\ \ldots, (p+s-1)e_{p+s+1}f_{k-s} = (e_1e_{p+s}f_{k-s} - e_{p+s}e_1f_{k-s}) = 0.$$

Hence $e_{k+p}f_1 = \cdots = e_{k+p}f_k = 0$ and

$$(k+p-1)e_{k+p+1}f_1 = (e_1e_{k+p}f_1 - e_{k+p}e_1f_1) = \cdots = (k+p-1)e_{k+p+1}f_k = 0.$$

Continuing these calculations we will have that $e_if_1 = \cdots = e_if_k = 0$ for all $i \ge k+p$. In particular $e_nf_1 = 0$. $\square$

Corollary 1. *Let $V = \langle f_1, f_2, \ldots, f_{n+1}\rangle$ be a $(n+1)$-dimensional graded thread W^+-module such that $e_nf_1 \neq 0$. Then for its defining set of constants $\alpha_i, i = 1, \ldots, n$, we have three possibilities:*

(a) no zeroes, $\alpha_i \neq 0,\ i = 1, \ldots, n$;
(b) the only one zero, $\exists! k,\ 1 \le k \le n,\ \alpha_k=0$;
(c) two neighboring zeroes, $\exists! k, 1\le k\le n-1, \alpha_k=\alpha_{k+1}=0$.

Following [3], we will consider a new basis of W^+:

$$\tilde{e}_1 = e_1, \quad \tilde{e}_i = 6(i-2)!e_i.$$

Now, we have in particular

$$[\tilde{e}_1, \tilde{e}_i] = \tilde{e}_{i+1},\ [\tilde{e}_2, \tilde{e}_3] = \tilde{e}_5,\ [\tilde{e}_2, \tilde{e}_5] = \frac{9}{10}\tilde{e}_7. \tag{1}$$

It was proved in [3] that the Lie algebra generated by two elements $\tilde{e}_1, \tilde{e}_2$ with the following two relations on them

$$\begin{array}{r} [\tilde{e}_2,[\tilde{e}_1,\tilde{e}_2]] = [\tilde{e}_1,[\tilde{e}_1,[\tilde{e}_1,\tilde{e}_2]]]\,, \\ [\tilde{e}_2,[\tilde{e}_1,[\tilde{e}_1,[\tilde{e}_1,\tilde{e}_2]]]] = \frac{9}{10}\,[\tilde{e}_1,[\tilde{e}_1,[\tilde{e}_1,[\tilde{e}_1,[\tilde{e}_1,\tilde{e}_2]]]]] \end{array} \tag{2}$$

is isomorphic to the positive part W^+ of the Witt algebra.

Hence, the defining relations (1) will give us the following set of polynomial equations

$$\begin{array}{ll} R^5_i: & ([\tilde{e}_2, \tilde{e}_3] - \tilde{e}_5)f_i = 0,\ \ i = 1, \ldots, n-5, \\ R^7_j: & ([\tilde{e}_2, \tilde{e}_5] - \frac{9}{10}\tilde{e}_7)f_j = 0,\ j = 1, \ldots, n-7. \end{array}$$

It is possible to write out the explicit expressions for R^5_i and R^7_j in terms of α_i, β_j. However rescaling $f_i \to \gamma_if_i$ we can make the constants α_i equal to one or to zero.

3 Graded Thread W^+-Modules of the Type $(1, 1, \ldots, 1)$.

Let us consider the case when all constants α_i of the defining set for a graded thread W^+-module are non-trivial and hence we may assume (after a suitable rescaling of the basis vectors $f_1, \ldots, f_{n+1}$) that

$$\tilde{e}_1 f_i = f_{i+1}, i = 1, \ldots, n, \quad \tilde{e}_2 f_j = b_j f_{j+2}, j = 1, \ldots, n-1.$$

Then the equations R_i^5, R_j^7 are read as

$$\begin{aligned} R_i^5: \quad & b_{i+3}(b_i - b_{i+1}) - b_i(b_{i+2} - b_{i+3}) = b_i - 3b_{i+1} + 3b_{i+2} - b_{i+3}, \\ R_j^7: \quad & b_{j+5}(b_j - 3b_{j+1} + 3b_{j+2} - b_{j+3}) - b_j(b_{j+2} - 3b_{j+3} + 3b_{j+4} - b_{j+5}) = \\ & = \frac{9}{10}(b_j - 5b_{j+1} + 10b_{j+2} - 10b_{j+3} + 5b_{j+4} - b_{j+5}), \\ & i = 1, \ldots, n-5,\ j = 1, \ldots, n-7. \end{aligned} \tag{3}$$

There is the relation between b_i and the initial structure constants α_i, β_j of our graded W^+-module V:

$$b_i = \frac{6\beta_i}{\alpha_i \alpha_{i+1}}, \quad i = 1, \ldots, i-2.$$

Consider a module $V_{\lambda,\mu}$. Recall that it has the defining set with $\alpha_i = i + \mu - 2\lambda$ and $\beta_j = j + \mu - 3\lambda$. We introduce new parameters

$$u = \mu - 3\lambda,\ v = \mu - 2\lambda.$$

Suppose that $v = \mu - 2\lambda \neq -1, -2, \ldots, -n$. Then $V_{\lambda,\mu}$ is of the type $(1, 1, \ldots, 1)$ and its coordinates $(b_1, b_2, \ldots, b_{n-1})$ are

$$b_i = 6\frac{(u+i)}{(v+i)(v+i+1)},\ i = 1, 2, \ldots, n-1.$$

Definition 2. An affine variety defined by the system of algebraic equations (3) in $\mathbb{K}^{n-1}$ is called the affine variety of $(n+1)$-dimensional graded thread W^+-modules of the type $(1, 1, \ldots, 1)$.

Theorem 1. *Let V be a $(n+1)$-dimensional graded thread W^+-module of the type $(1, 1, \ldots, 1)$ and $n \geq 9$, i.e. W^+-module defined by its basis and defining set of relations*

$$\begin{gathered} V = \langle f_1, f_2, \ldots, f_{n+1} \rangle, \\ e_1 f_i = f_{i+1},\ i = 1, 2, \ldots, n; \\ e_2 f_j = b_j f_{j+2},\ j = 1, 2, \ldots, n-1, \end{gathered}$$

Then, V is isomorphic to the one and only one W^+-module from the list below.

- $V_{\lambda,\mu}(n+1), \quad \mu - 2\lambda \neq -1, -2, \ldots, -n.$
- $C_{1,x}(n+1), \quad b_1 = b_2 = \cdots = b_{n-1} = x;$

- $V^t_{-2,-3}(n+1),\ t \neq 4, \quad b_1 = t, b_i = \frac{6(i+3)}{(i+1)(i+2)}, i = 2, \ldots, n-1;$
- $V^t_{1,3-n}(n+1),\ t \neq 4, \quad b_i = -\frac{6(n-i)}{(n-i-2)(n-i-1)}, i = 1, \ldots, n-2, b_{n-1} = -t;$
- $V^t_{0,-1}(n+1),\ t \neq 6, \quad b_1 = t, b_i = \frac{6}{i}, i = 2, \ldots, n-1;$
- $V^t_{-1,-2-n}(n+1),\ t \neq 6, \quad b_i = -\frac{6}{(n-i)}, i = 1, \ldots, n-2, b_{n-1} = t;$

Remark 3. (1) A one-parametric family $V^t_{-2,-3}(n+1)$ of graded thread W^+-modules is a linear deformation of $V_{-2,-3}(n+1)$. Moreover $V^4_{-2,-3}(n+1) = V_{-2,-3}(n+1)$. A module $V^t_{1,3-n}(n+1)$ is dual to $V^t_{-2,-3}(n+1)$.

(2) Family $V^t_{0,-1}(n+1)$ is one-parametric linear deformation of $V_{0,-1}(n+1)$ and $V^6_{0,-1}(n+1) = V_{0,-1}(n+1)$. $V^t_{1,-2-n}(n+1)$ is the dual module to $V^t_{0,-1}(n+1)$.

(3) $C^*_{1,x}(n+1) = C_{1,-x}(n+1)$;

Proof. We prove the Theorem by induction on dimension $\dim V$. The equation R^5_1 appears first time for a 5-dimensional graded thread module. In dimension 6 we have two relations R^5_1, R^5_2 and coordinates $(b_1, b_2, \ldots, b_5)$ of an arbitrary 7-dimensional graded thread W^+-module V of the type $(1,1,\ldots,1)$ satisfies three relations R^5_1, R^5_2, R^7_1 respectively.

We start with the classification of 8-dimensional graded thread W^+-modules.

Lemma 3. *Consider an affine variety M of 8-dimensional graded thread W^+-modules of the type $(1,1,\ldots,1)$ defined by the following system of quadratic equations in $\mathbb{K}^6$:*

$$\begin{aligned}
&b_4(b_1-b_2) - b_1(b_3-b_4) = b_1 - 3b_2 + 3b_3 - b_4,\\
&b_5(b_2-b_3) - b_2(b_4-b_5) = b_2 - 3b_3 + 3b_4 - b_5,\\
&b_6(b_3-b_4) - b_3(b_5-b_6) = b_3 - 3b_4 + 3b_5 - b_6,\\
&b_6(b_1-3b_2+3b_3-b_4) - b_1(b_3-3b_4+3b_5-b_6) = \tfrac{9}{10}(b_1-5b_2+10b_3-10b_4+5b_5-b_6),
\end{aligned} \tag{4}$$

then the variety M can be decomposed as the union of the following two- and one-parametric algebraic subsets:

$M_1:\ b_i = 6\frac{(u+i)}{(v+i)(v+i+1)},\ i = 1, 2, \ldots, 6,\ u \neq v, u \neq v+1,\ v \neq -1, -2, \ldots, -7;$
$M^0_1:\ b_i = \frac{6}{v+i},\ i = 1, 2, \ldots, 6,\ v \neq -1, -2, \ldots, -6;$
$M_2:\ \begin{array}{l} b_1 = \frac{5xy-17x+10y+2}{5y-9},\ b_2 = x,\ b_3 = y,\ b_4 = y - \frac{2}{5},\ b_5 = \frac{1}{5}\frac{5xy+3x-6}{2x-y+1}, \\ b_6 = \frac{5xy^2-2xy-22y+10y^2+21x-12}{(2x-y+1)(5y+7)},\ y \neq \frac{9}{5}, -\frac{7}{5}, 2x-y+1 \neq 0 \end{array};$
$M_3:\ b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = t;$
$M^{\pm}_4:\ \begin{array}{l} b_1 = 12 \pm 3\sqrt{19}, b_2 = -\frac{2}{5} \pm \frac{1}{5}\sqrt{19}, b_3 = \frac{1}{5} \pm \frac{2}{5}\sqrt{19}, b_4 = -\frac{1}{5} \pm \frac{2}{5}\sqrt{19}, \\ b_5 = \frac{2}{5} \pm \frac{1}{5}\sqrt{19} + t, b_6 = -12 \pm 3\sqrt{19} + t(\pm\frac{4}{3}\sqrt{19} - \frac{13}{3}). \end{array};$
$M^-_5:\ b_1 = -\frac{27}{28}, b_2 = -\frac{8}{7}, b_3 = -\frac{7}{5}, b_4 = -\frac{9}{5}, b_5 = -\frac{5}{2}, b_6 = t;$
$M^+_5:\ b_1 = t, b_2 = \frac{5}{2}, b_3 = \frac{9}{5}, b_4 = \frac{7}{5}, b_5 = \frac{8}{7}, b_6 = \frac{27}{28};$
$M^+_6:\ b_1 = t, b_2 = \frac{6}{2}, b_3 = \frac{6}{3}, b_4 = \frac{6}{4}, b_5 = \frac{6}{5}, b_6 = \frac{6}{6};$
$M^-_6:\ b_1 = -\frac{6}{6}, b_2 = -\frac{6}{5}, b_3 = -\frac{6}{4}, b_4 = -\frac{6}{3}, b_5 = -\frac{6}{2}, b_6 = -t;$

Proof. Denote $b_2 = x, b_3 = y, b_4 = z$ and rewrite after that first two equations of the system (4)

$$\begin{cases} b_1(2z-y-1) = xz - 3x + 3y - z, \\ b_5(2x-y+1) = xz + x - 3y + 3z. \end{cases} \tag{5}$$

Then, we multiply the third equation of (4) by $(2x-y+1)$ and exclude b_5

$$(2x-y+1)(2y-z+1)b_6 = xyz - 4y^2 + 3yx + 6yz - 8y - 3xz + 6z + 3x.$$

Finally, we multiply the last equation of (4) by $(2z-y-1)(2x-y+1)(2y-z+1)$ and exclude b_1, b_5, b_6. We will get the following fifth-order equation of unknowns x, y, z:

$$\left(z - y - \frac{2}{5}\right) F(x,y,z) = 0,$$

where

$$F(x,y,z) = y^2(z-6)(x+6) + y(x+z)(xz+3z-3x+36) + 3xz(4x-4z-xz) - 9(x+z)^2. \tag{6}$$

First of all we are going to study an algebraic variety $M_F \subset \mathbb{K}^3$ defined by the equation $F(x,y,z)=0$. Consider the mapping of uniformization

$$\begin{array}{l} f : \mathbb{K}^2 \backslash \{(u,v), v = 0, -1, -2, -3\} \to \mathbb{K}^3 : \\ f(u,v) = \left(\frac{6u}{v(v+1)}, \frac{6(u+1)}{(v+1)(v+2)}, \frac{6(u+2)}{(v+2)(v+3)} \right). \end{array}$$

Proposition 1. *The variety $M_F \subset \mathbb{K}^3$ is the union of Imf and three lines l_1, l_2, l_3 defined by linear equations*

$$l_1 : x = y = z; \quad l_2 : \begin{cases} y = 3, \\ z = 2 \end{cases}, \quad l_3 : \begin{cases} x = -2, \\ y = -3, \end{cases}$$

Proof. Let a point $(x,y,z) \in Imf$. It means that for some $(u,v), v \neq 0, -1, -2, -3$ we have

$$\begin{cases} 6u = x(v^2+v), \\ 6u + 6 = y(v^2+3v+2), \\ 6u + 12 = z(v^2+5v+6). \end{cases} \tag{7}$$

Consider (7) as a system of equations with respect to unknowns u, v. For $z \neq x$ it is equivalent to

$$\begin{cases} 6u = x(v^2+v), \\ (x-z)v^2 + (x-5z)v + 12 - 6z = 0, \\ ((3y-x)(z-x) - (5z-x)(y-x))\, v = (6z-12)(y-x) - (2y-6)(z-x). \end{cases} \tag{8}$$

(1) Let $(3y-x)(z-x) - (5z-x)(y-x) = 2(2xz - y(x+z)) \neq 0$, then substituting

$$v = \frac{2yz - 3zx - 6y + 3x + yx + 3z}{2xz - y(x+z)}$$

in the second equation of (8) we get

$$\frac{(x-z)F(x,y,z)}{(y(x+z) - 2xz)^2} = 0,$$

where the polynomial $F(x, y, z)$ is defined by (6). Hence a point (x, y, z) of the surface M_F with $x \neq z$ and $y(x{+}z) \neq 2xz$ is in the image Imf and the corresponding parameters u, v are determined uniquely.

(2) Consider a point $(x, y, z) \in M_F$ such that

$$\begin{cases} (6z - 12)(y - x) - (2y - 6)(y - x) = 0, \\ 2xz - y(x + z) = 0. \end{cases} \tag{9}$$

This system can be rewritten in a following way

$$\begin{cases} y(z - 6) = xz - 3x - 3z, \\ y(x + z) = 2xz. \end{cases}$$

Substitute $y = \frac{2xz}{x+z}$ ($x + z = 0$ implies $x = z = 0$) in the first equation. We obtain $(z{-}x)(xz{+}3z{-}3x){=}0$, hence if $z \neq x$ then

$$x = \frac{3z}{3 - z},\ y = \frac{6z}{6 - z}.$$

Changing parameter $z = \frac{6}{t+2}$ we see that the set of solutions of (9) coincides with the curve

$$\gamma(t) = \left(\frac{6}{t}, \frac{6}{t + 1}, \frac{6}{t + 2}\right). \tag{10}$$

Parameters u, v for a point $\left(\frac{6}{t}, \frac{6}{t+1}, \frac{6}{t+2}\right)$ of γ are determined not uniquely: $u = v = t$ or $u = v + 1 = t + 2$.

Now, let us consider the case $x = z$. Then, the system (8) is equivalent to the following one

$$\begin{cases} 6u = x(v^2 + v), \\ (y - x)v^2 + (3y - x)v + 2y - 6 = 0, \\ 2xv = 6 - 3x. \end{cases} \tag{11}$$

Substituting v by $\frac{6-3x}{2x}$ in the second equation of (11), we'll get

$$\frac{y(36 - x^2) - 36x - 3x^3}{4x^2} = 0.$$

It follows from the last equation that $y = \frac{36x+3x^3}{36-x^2}, x \neq 6$.

On the another hand, the square equation (with respect to y, we assume also that $x \neq \pm 6$)

$$F(x, y, x) = y^2(x^2{-}36) + 2x(x^2{+}36)y - 3x^4 - 36x^2 = 0$$

has two roots $y = x$ and $y = \frac{36x+3x^3}{36-x^2}$. One has to remark also that if $x = z = \pm 6$, then $y = \pm 6$.

Hence, an arbitrary point $P = (x, y, z)$ of the surface M_F with $x = z \neq y$ also belongs to the image Imf. We conclude with a remark that the system (7) never has the solutions with $v = -1, -2$, but it has the solutions with $v = 0$ and $v = -3$. More precisely, if $v = 0$ then it follows from the first equation of (7) that $u = 0$ and this is the case for $y = 3, z = 2$ and arbitrary x. We get the line l_2.

Analogously the case $v = -3$ implies $u = -2$ and $y = -3, x = -2$. The corresponding set of solutions is l_3. □

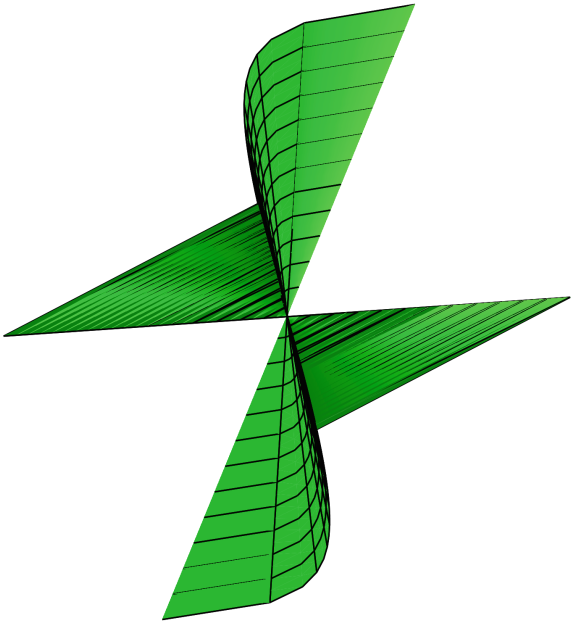

Fig. 1. The surface M_F in $\mathbb{R}^3$.

Remark 4. The polynomial $F(x, y, z)$ has the degree two with respect to each variable x, y, z (thinking other two are parameters). One can verify directly that the curve $\gamma(t)$ defined by (10) coincides with the set of singular points of M_F (it follows from the proof of our proposition). Also one can see that M_F has an involution $\sigma_F : M_F \to M_F$:

$$\sigma_F : (x, y, z) \to (-z, -y, -x).$$

More precisely,

$$\sigma_F(f(u, v)) = f(-u-2, -v-4),\ \sigma_F(l_1) = l_1,\ \sigma_F(l_2) = l_3.$$

Now, we come to the following natural question.

Does an arbitrary point $(x, y, z) \in M_F$ *correspond to some solution* (b_1, x, y, z, b_5, b_6) *of the initial system* (4)*?*

Before answering this question, we state a few preliminary remarks.

Proposition 2. *It exists the unique solution* $(b_1, b_2, \ldots, b_6)$ *of the system (4) with* $b_2 = b_3 = b_4 = t$ *and it is* $b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = t$.

Proof. If $b_2 = b_3 = b_4 = t$, then the first two equations of (4) can be rewritten as

$$b_1(t-1) = t^2 - t, \quad b_5(t+1) = t^2 + t.$$

If $t \neq \pm 1$ then $b_1 = b_5 = t$ and $b_6 = \frac{t^3+2t^2+t}{(t+1)^2} = t$.

If $t = 1$, then $b_5 = b_6 = 1$ and the fourth equation of (4) will look in the following way:

$$(b_1 - 1) - \frac{9}{10}(b_1 - 1) = 0.$$

Hence, $b_1 = 1$. The case $t = -1$ is studied analogously. □

Proposition 3. *There are no solutions* $(b_1, b_2, \ldots, b_6)$ *of the system (4) such that:*

(a) $b_3 = 3, b_4 = 2$*;*
(b) $b_3 = -3, b_2 = -2$.

Proof. Solving first three equations of (4) with $b_3 = 3, b_4 = 2$ one get

$$b_2 = 7, b_5 = \frac{3}{2}, b_6 = \frac{6}{5}.$$

But the fourth equation of (4) looks as $0 \cdot b_6 = -\frac{339}{10}$.

If $b_3 = -3$ and $b_2 = -2$ we obtain $b_1 = -\frac{3}{2}, b_4 = -7, b_6 = -9$ and the last equation of (4) will be inconsistent with respect to b_5. □

Now, we are going to study the same question for the points of Imf. Namely, let suppose that for some choice of $u, v, v \neq 0, -1, -2, -3$, we have

$$b_2 = \frac{6u}{v(v+1)}, \; b_3 = \frac{6(u+1)}{(v+1)(v+2)}, \; b_4 = \frac{6(u+2)}{(v+2)(v+3)}.$$

Then the first three equations of the system (4) we rewrite as

$$\begin{aligned} b_5 \frac{(v+4)(6u+v^2-v)}{v(v+1)(v+2)} &= 6\frac{(u+3)(6u+v^2-v)}{v(v+1)(v+2)(v+3)}, \\ b_1 \frac{(v-1)(6u-v^2-7v)}{(v+1)(v+2)(v+3)} &= 6\frac{(u-1)(6u-v^2-7v)}{v(v+1)(v+2)(v+3)}, \\ b_6 \frac{(v+5)(6u+v^2+v+6)}{(v+1)(v+2)(v+3)} &= b_5 \left(\frac{6(u+1)}{(v+1)(v+2} + 3 \right) - 6\frac{2uv+5v+3}{(v+1)(v+2)(v+3)}. \end{aligned} \tag{12}$$

First of all, let us study a generic case.

Proposition 4. *Let $v \neq 0, -1, -2, -3$ and moreover*

$$(v{+}4)(6u{+}v^2{-}v) \neq 0,\ (v{-}1)(6u{-}v^2{-}7v) \neq 0,\ (v{+}5)(6u{+}v^2{+}v{+}6) \neq 0.$$

Then, there exists the only one solution (b_1, b_5, b_6) of the system (12)

$$b_1 = \frac{6(u{-}1)}{(v{-}1)v},\ b_5 = \frac{6(u{+}3)}{(v{+}3)(v{+}4)},\ b_6 = \frac{6(u{+}4)}{(v{+}4)(v{+}5)}.$$

Proof. Direct verification. □

Proposition 5. *Let $v \neq 0, -1, -2, -3$. Then, if one of the three given expressions*

$$6u + v^2 - v,\ 6u - v^2 - 7v,\ 6u + v^2 + v + 6,$$

is equal to zero, the rest of them are non trivial.

Now we assume that $v \neq 1, 0, -1, -2, -3, -4, -5$.

(1) $6u - v^2 - 7v = 0$. Then, from the first and the third equations of (4) we have

$$b_5 = \frac{6(u{+}3)}{(v{+}3)(v{+}4)},\ b_6 = \frac{6(u{+}4)}{(v{+}4)(v{+}5)}.$$

The fourth equation of (4) will look as

$$\frac{1}{10}b_1\frac{v(v{-}1)(v{-}2)(v{-}3)}{(v{+}2)(v{+}3)(v{+}4)(v{+}5)} = \frac{1}{10}\frac{(v{-}2)(v{-}3)(v^2+7v-6)}{(v{+}2)(v{+}3)(v{+}4)(v{+}5)}.$$

If $v \neq 2, 3$ then $b_1 = \frac{v^2+7v-6}{v(v-1)} = \frac{6(u-1)}{v(v-1)}$ as in the generic situation.

(a) Now, let $v = 2$ then $u = \frac{v^2+7v}{6} = 3$ and we have

$$b_2 = 3, b_3 = 2, b_4 = \frac{3}{2}, b_5 = \frac{6}{5}, b_6 = 1.$$

The component b_1 can take an arbitrary value t.

(b) In the case $v = 3$, analogously, we obtain $u = 5$ and

$$b_2 = \frac{5}{2}, b_3 = \frac{9}{5}, b_4 = \frac{7}{5}, b_5 = \frac{8}{7}, b_6 = \frac{27}{28}.$$

We obtain another one-parametric family of solutions for (4):

$$(b_1, b_2, \ldots, b_6) = \left(t, \frac{5}{2}, \frac{9}{5}, \frac{7}{5}, \frac{8}{7}, \frac{27}{28}\right).$$

(2) $6u + v^2 - v = 0$. Then $b_1 = \frac{6(u-1)}{(v-1)v}$ and the third equation of (4) will look as

$$b_6(v{+}5) = b_5(v^2{+}4v{+}3) - v^2 + 4v + 3.$$

Expressing b_6 and rewriting the last equation of (4) we will have

$$-\frac{1}{10}b_5\frac{(v{+}4)(v{+}3)(v^2{+}8v{-}3)}{(v{+}1)v(v{-}1)}=\frac{1}{10}\frac{(v^2{+}8v{-}3)(v^2{-}v{-}18)}{v(v{-}1)(v{+}1)}.$$

If $v^2+8v-3\neq 0$ then b_5 is determined uniquely and hence we have

$$b_5=\frac{6(u{+}3)}{(v{+}3)(v{+}4)},\ b_6=\frac{6(u{+}4)}{(v{+}4)(v{+}5)}.$$

If $v^2+8v-3=0$, i.e. $v=-4\pm\sqrt{19}$, then the component b_5 can take an arbitrary values and then we have $u=-\frac{13}{2}\pm\frac{3}{2}\sqrt{19}$ and

$$b_1=12\pm 3\sqrt{19},\ b_2=-\frac{2}{5}\pm\frac{1}{5}\sqrt{19},\ b_3=\frac{1}{5}\pm\frac{2}{5}\sqrt{19},$$
$$b_4=-\frac{1}{5}\pm\frac{2}{5}\sqrt{19}, b_5=t, b_6=-\frac{46}{3}\pm\frac{10\sqrt{19}}{3}+t\left(\pm\frac{4}{3}\sqrt{19}-\frac{13}{3}\right).$$

After a parameter change $t\rightarrow t+\frac{2}{5}\pm\frac{1}{5}\sqrt{19}$ we will obtain the final version.

(3) $6u+v^2+v+6=0$. Then $b_1=\frac{6(u-1)}{(v-1)v}$ and $b_5=\frac{6(u+3)}{(v+3)(v+4)}$. The fourth equation of the system (4) will look as

$$-\frac{1}{10}b_6\frac{(v{+}5)(v{+}4)(v{+}7)(v{+}6)}{(v{+}1)(v{+}2)(v{-}1)v}=\frac{1}{10}\frac{(v{+}7)(v{+}6)(v^2{-}v{-}18)}{v(v{-}1)(v{+}1)(v{+}2)}.$$

If $v\neq -6,-7$, then $b_6=\frac{6(u+4)}{(v+4)(v+5)}$.

For $v=-6, u=-6$, we get an one-parametric family

$$(b_1,b_2,\dots,b_6)=\left(-1,-\frac{6}{5},-\frac{6}{4},-\frac{6}{3},-\frac{6}{2},t\right).$$

The case $v=-7, u=-8$ corresponds to another one line of solutions:

$$(b_1,b_2,\dots,b_6)=\left(-\frac{27}{28},-\frac{8}{7},-\frac{7}{5},-\frac{9}{5},-\frac{5}{2},t\right).$$

(4) Let $v=-4$. Then, the first equation implies that $u=-3$ or $u=-\frac{10}{3}$.

(a) Let $v=-4, u=-3$. Then, $b_1=-\frac{6}{5}, b_2=-\frac{3}{2}, b_3=-2, b_4=-3$. The third equation gives $b_5=-7$, but the fourth equation becomes inconsistent with respect to b_6.

(b) Let $v=-4, u=-\frac{10}{3}$, then third and fourth equations of (4) look as

$$\begin{cases}\frac{1}{3}b_6-\frac{2}{3}b_5-\frac{29}{3}=0,\\ \frac{3}{10}b_6-\frac{3}{5}b_5-\frac{2629}{300}=0,\end{cases}$$

and this system is inconsistent with respect to b_5, b_6.

(5) If $v = 1$, then the second equation of (4) implies that $u = 1$ or $u = \frac{4}{3}$.

(a) The case $v = u = 1$ corresponds to the family

$$(b_1, b_2, \ldots, b_6) = \left(t, 3, 2, \frac{3}{2}, \frac{6}{5}, 1\right).$$

(b) If $v = 1, u = \frac{4}{3}$, the fourth equation of (4) degenerates to $-\frac{1}{900} = 0$ and hence the system (4) is inconsistent.

(6) Let $v = 5$, the first equation of (4) gives us $b_5 = 3(u{+}3)$, the third equation after substitution $v = -5, b_5 = 3(u{+}3)$ will look like

$$\frac{3}{2}u^2 + \frac{25}{2}u + 26 = 0.$$

It has the roots $u = -\frac{13}{3}, -4$.

(a) The case $v = -5, u = -4$ corresponds to the family that we already obtained:

$$(b_1, b_2, \ldots, b_6) = \left(-1, -\frac{6}{5}, -\frac{6}{4}, -\frac{6}{3}, -\frac{6}{2}, t\right).$$

(b) If $v = -5, u = -\frac{13}{3}$ then $b_1 = -\frac{16}{15}, b_2 = -\frac{13}{10}, b_3 = -\frac{5}{3}, b_4 = -\frac{7}{3}, b_5 = -4$ and the fourth equation is inconsistent with respect to b_6.

We studied the solutions of the main system (4) that correspond to the points of the algebraic variety $M_F = \{F(x, y, z) = 0\}$. Now, we will consider the case

$$b_2 = x, \quad b_3 = y, \quad b_4 = y - \frac{2}{5} = z.$$

Then, one can rewrite the first three equations of (4) as

$$\begin{cases} (y{-}\frac{9}{5})b_1 = xy + 2y - \frac{17}{5}x + \frac{2}{5}, \\ (2x{-}y{+}1)b_5 = xy + \frac{3}{5}x - \frac{6}{5}, \\ (y{+}\frac{7}{5})b_6 = b_5(y{+}3) - 2y + \frac{6}{5}. \end{cases} \tag{13}$$

Proposition 6. *There are no solutions of (13) with* $y = \frac{9}{5}, -\frac{7}{5}$.

Proof. Direct calculations. □

Proposition 7. *Let* $2x - y + 1 = 0$*. Then, the system (13) is consistent if and only if*

$$x = -\frac{2}{5} \pm \frac{1}{5}\sqrt{19},\ y = \frac{1}{5} \pm \frac{2}{5}\sqrt{19},\ z = y - \frac{2}{5} = -\frac{1}{5} \pm \frac{2}{5}\sqrt{19}, b_5 = t.$$

Proof. Indeed $2x - y + 1 = 0$ implies $xy + \frac{3}{5}x - \frac{6}{5} = 0$ that is equivalent to $2x^2 + \frac{8}{5}x - \frac{6}{5} = 0$. The roots of this square equation will give us the values given above. Now, one can express $b_1 = 12 \pm 3\sqrt{19}$, take $b_5 = t$, then express b_6 in terms of b_5 taking into account third equation. Finally, we obtain

$$b_6 = -\frac{46}{3} \pm \frac{10\sqrt{19}}{3} + t\left(\pm\frac{4}{3}\sqrt{19} - \frac{13}{3}\right).$$

That corresponds to the family of solutions that we have already obtained. □

Now, it is easy to see that in the generic situation $(2x-y+1 \neq 0)$ we have

$$b_1 = \frac{5xy-17x+10y+2}{5y-9},\ b_2 = x,\ b_3 = y,\ b_4 = y - \frac{2}{5},$$
$$b_5 = \frac{1}{5}\frac{5xy+3x-6}{2x-y+1},\ b_6 = \frac{5xy^2-2xy-22y+10y^2+21x-12}{(2x-y+1)(5y+7)}.$$

The last statement concludes the proof of the theorem.

Remark 5. We shifted the arguments u, v in our answer for M_1, for instance we considered $b_2 = \frac{6u}{v(v+1)}$ in our proof and now $b_2 = \frac{6(u+2)}{(v+2)(v+3)}$. One has to point out some properties of the subsets M_i:

- M_3 does not intersect other subsets M_i;
- $P(u,v) \in M_1$ belongs to M_2 if and only if $u = \frac{1}{30}(v+3)(v+4)(v+5) + \frac{1}{2}(v-3)$;
- M_1^0 intersects M_2 at $v = \frac{-7\pm\sqrt{61}}{2}$;
- $M_4^{\pm}$ intersects M_1 at $t = -\frac{16}{15} \pm \frac{68}{285}\sqrt{19}$ ($P_4^{\pm} = P(-\frac{9}{2} \pm \frac{3}{2}\sqrt{19}, -2 \pm \sqrt{19})$ and does not intersect other subsets M_i;
- $M_5^{\pm}$ intersects M_1 at $t = \pm 4$ (the points $P_5^{+} = P(3,1)$ and $P_5^{-} = P(-10,-9)$ respectively);
- $M_5^{\pm}$ intersects M_2 at $t = 0$;
- $M_6^{\pm}$ intersects M_1^0 at $t = \pm 6$ and $v = 0$ and does not intersect other subsets;

One can also remark that there exists an involution $\sigma : M \to M$

$$\sigma : (b_1, b_2, \ldots, b_5, b_6) \to (-b_6, -b_5, \ldots, -b_2, -b_1)$$

with the properties:

- M_1 is invariant with respect to σ: $\sigma(P(u,v)) = P(-u-7, -v-8)$;
- $\sigma(M_1^0) = M_1^0$;
- $\sigma(M_2 \cup M_4^{\pm}) = M_2 \cup M_4^{\pm}$;
- $\sigma(M_3) = M_3$;
- $\sigma(M_5^{-}) = M_5^{+}$;
- $\sigma(M_6^{-}) = M_6^{+}$;

Corollary 2. *The affine variety of* 9*-dimensional graded thread* W^+*-modules*

$$V = \langle f_1, f_2, \ldots, f_9 \rangle,$$
$$e_1 f_i = f_{i+1},\ i = 1, 2, \ldots, 8;$$
$$e_2 f_j = b_j f_{j+2},\ j = 1, 2, \ldots, 7,$$

can be parametrized by means of two- and one-parametric algebraic subsets

$\tilde{M}_1 : b_i = 6\frac{(u+i)}{(v+i)(v+i+1)},\ i = 1, 2, \ldots, 6, 7,\ u \neq v, u \neq v+1, v \neq -1, -2, \ldots, -7, -8$*;*
$\tilde{M}_1^0 : b_i = \frac{6}{v+i},\ i = 1, 2, \ldots, 6, 7,\ v \neq -1, -2, \ldots, -6, -7$*;*

$$\tilde{M}_2: \begin{array}{c} b_1{=}\frac{(y-\frac{3}{5})(y+\frac{3}{5})(y-2)}{(y-\frac{9}{5})(y-\frac{7}{5})}, b_2{=}\frac{(y-\frac{8}{5})(y+\frac{3}{5})}{(y-\frac{9}{5})}, b_3{=}y{+}\frac{2}{5}, b_4{=}y, b_5{=}y{-}\frac{2}{5}, \\ b_6{=}\frac{(y+\frac{8}{5})(y-\frac{3}{5})}{(y+\frac{9}{5})}, b_7{=}\frac{(y-\frac{3}{5})(y+\frac{3}{5})(y+2)}{(y+\frac{9}{5})(y+\frac{7}{5})} \end{array};$$

$\tilde{M}_3:\ b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = t;$
$\tilde{M}_5^-:\ b_1 = -\frac{5}{6}, b_2 = -\frac{27}{28}, b_3 = -\frac{8}{7}, b_4 = -\frac{7}{5}, b_5 = -\frac{9}{5}, b_6 = -\frac{5}{2}, b_7 = t;$
$\tilde{M}_5^+:\ b_1 = t, b_2 = \frac{5}{2}, b_3 = \frac{9}{5}, b_4 = \frac{7}{5}, b_5 = \frac{8}{7}, b_6 = \frac{27}{28}, b_7 = \frac{5}{6};$
$\tilde{M}_6^+:\ b_1 = t, b_2 = \frac{6}{2}, b_3 = \frac{6}{3}, b_4 = \frac{6}{4}, b_5 = \frac{6}{5}, b_6 = \frac{6}{6}, b_7 = \frac{6}{7};$
$\tilde{M}_6^-:\ b_1 = -\frac{6}{7}, b_2 = -\frac{6}{6}, b_3 = -\frac{6}{5}, b_4 = -\frac{6}{4}, b_5 = -\frac{6}{3}, b_6 = -\frac{6}{2}, b_7 = -t.$

Proof. The families $M_4^{\pm}$ do not survive to the dimension 9, $\tilde{M}_2$ became an one-parametric family instead of two-parametric M_2. $\tilde{M}_2$ intersects $\tilde{M}_1$ at $y = \pm\frac{2}{5}\sqrt{21}$. □

We already classified 9-dimensional graded thread W^+-modules. Let V be a 10-dimensional W^+-module with the basis $f_1, \dots, f_{10}$. Consider its quotient module $\hat{V} = V/\langle f_{10}\rangle$ and its submodule $\tilde{V} = \langle f_2, \dots, f_{10}\rangle$. Both of them are 9-dimensional graded thread modules of the type $(1, 1, \dots, 1)$ with the defining sets $b_1, \dots, b_7$ and $b_2, \dots, b_8$ respectively.

Only two points $y = \pm\frac{2}{5}\sqrt{21}$ from $\tilde{M}_2$ survive to dimension 10. These points are in the intersection $\tilde{M}_1 \cap \tilde{M}_2$.

Let $(b_1, b_2, \dots, b_{n-2}, b_{n-1})$ be coordinates (defining set) of a $(n+1)$-dimensional graded thread W^+-module $V = \langle f_1, f_2, \dots, f_n, f_{n+1}\rangle$ of the type $(1, 1, \dots, 1)$ then its subsets $(b_1, b_2, \dots, b_{n-2})$ and $(b_2, \dots, b_{n-2}, b_{n-1})$ are coordinates of n-dimensional quotient $V/\langle f_{n+1}\rangle$ and submodule $\langle f_2, \dots, f_n, f_{n+1}\rangle$ respectively. Both of them are n-dimensional graded thread W^+-modules of the type $(1, 1, \dots, 1)$ and we can apply the induction hypothesis.

For instance, let $(b_2, b_3, \dots, b_{n-2}, b_{n-1}) = (t, 3, \dots, \frac{6}{n-2}, \frac{6}{n-1})$ then $(b_1, t, 3, \dots, \frac{6}{n-2})$ have also to be present in the Table below and it is impossible. On the another hand, if $(b_1, b_2, \dots, b_{n-2}) = (t, 3, \dots, \frac{6}{n-2})$, then we have to find the set $(b_2, \dots, b_{n-1}) = (\frac{6}{2}, \dots, \frac{6}{n-2}, b_{n-1})$ in the Table. It can occure if and only if $b_{n-1} = \frac{6}{n-1}$.

4 Graded Thread Modules of the Type $(1, \dots, 1, 0, 1, \dots, 1)$.

Now, we consider the case, when one and only one of the defining constants α_i vanishes.

$$\exists!\, k, 1 \le k \le n,\ \alpha_k = 0, \text{ i.e. } \tilde{e}_1 f_k = 0,\ \tilde{e}_1 f_j \neq 0, j \neq k, 1 \le j \le n.$$

A graded module of this type is called "décousu" in [3].

Theorem 2. *Let V be a $(n{+}1)$-dimensional, $n \ge 16$, indecomposable graded thread W^+-module of the type $(1, \dots, 1, 0, 1, \dots, 1)$, i.e. there exists a basis $f_i, i = 1, \dots, n{+}1$, of V and $\exists! k, 1 \le k \le n$, such that*

$$\tilde{e}_i f_j \in \langle f_{i+j}\rangle, i+j \le n+1,\ \tilde{e}_i f_j = 0, i+j > n+1,$$
$$\tilde{e}_1 f_k = 0,\ \tilde{e}_1 f_i \neq 0,\ i = 1, \dots, k{-}1, k{+}1, \dots, n;$$

Table 1. Graded thread W^+-modules of the type $(1, 1, \ldots, 1), \dim V = n + 1 \geq 10$.

Module	b_1	b_2	...	b_i	...	b_{n-2}	b_{n-1}
$V_{\lambda,\mu}(n+1)$, $u = \mu-3\lambda$, $v = \mu-2\lambda$, $v \neq -1, \ldots, -n$,	$\frac{6(u+1)}{(v+1)(v+2)}$	$\frac{6(u+2)}{(v+2)(v+3)}$	...	$\frac{6(u+i)}{(v+i)(v+i+1)}$	...	$\frac{6(u+n-2)}{(v+n-2)(v+n-1)}$	$\frac{6(u+n-1)}{(v+n-1)(v+n)}$
$C_{1,x}(n+1)$	x	x	...	x	...	x	x
$V^t_{-2,-3}(n+1)$, $t \neq 4$	t	$\frac{5}{2}$	...	$\frac{6(i+3)}{(i+1)(i+2)}$	...	$\frac{6(n+1)}{(n-1)n}$	$\frac{6(n+2)}{n(n+1)}$
$V^t_{1,3-n}(n+1)$, $t \neq 4$	$-\frac{6(n+2)}{n(n+1)}$	$-\frac{6(n+1)}{(n-1)n}$	...	$-\frac{6(n-i)}{(n-i-2)(n-i-1)}$	...	$-\frac{5}{2}$	$-t$
$V^t_{0,-1}(n+1)$, $t \neq 6$	t	3	...	$\frac{6}{i}$	...	$\frac{6}{(n-2)}$	$\frac{6}{(n-1)}$
$V^t_{-1,-2-n}(n+1)$, $t \neq 6$	$-\frac{6}{(n-1)}$	$-\frac{6}{(n-2)}$	...	$-\frac{6}{(n-i)}$	...	-3	$-t$

Then, V is isomorphic to one and only one module from the following list

(A) $(n+1)$-dimensional quotients of $V_{\lambda,2\lambda-k}, \lambda \neq 0, -1$,

- $V_{\lambda,2\lambda-k}(n+1), \lambda \neq 0, -1, \ 1 \leq k \leq n$;

(B) linear deformations of $(n+1)$-dimensional quotients $V_{-1,-k-2}$ and $V_{0,-k}$

- $V^t_{-1,-k-2}(n+1), \ 1 \leq k < n-1, \ t \in \mathbb{K}$; *defined by*

$$e_i f_j = \begin{cases} (j+i-k-1) f_{i+j}, j \neq k+1, i+j \leq n+1, \\ 0, \quad i+j > n+1; \end{cases}, \qquad e_i f_{k+1} = \begin{cases} i(t(i-1)-i+2) f_{i+k+1}, i \leq n-k, \\ 0, \quad i > n-k. \end{cases} \tag{14}$$

- $V^t_{0,-k}(n+1), 2 < k \leq n, \ t \in \mathbb{K}$; *it is dual to the W^+-module* $V^{-t}_{-1,-k-2}(n+1)$

$$V^t_{0,-k}(n+1) = V^{-t*}_{-1,-k-2}(n+1);$$

(C) degenerate cases for $k = 1, 2, n-1, n$.

- $V_{0,-2}(n+1), \ k = 2$;
- $V_{-1,-4}(n+1) = V^*_{0,-2}(n+1), \ k = n-1$;
- $\tilde{V}_{0,-1}(n+1), \ k = 1$, *defined by*

$$e_i f_j = \begin{cases} (j-1) f_{i+j}, j \geq 2, i+j \leq n+1, \\ 0, j \geq 2, i+j > n+1, \end{cases} \quad e_i f_1 = \begin{cases} (i-1) f_{i+1}, 1 \leq i \leq n, \\ 0, \ i > n. \end{cases}; \tag{15}$$

- $\tilde{V}_{-1,-2-n}(n+1) = \tilde{V}^*_{0,-1}(n+1), \ k = n$;
- $\tilde{V}_{-2,-3}(n+1), \ k = 1$, *defined by*

$$e_i f_j = \begin{cases} (j+2i-1) f_{i+j}, j \geq 2, i+j \leq n+1, \\ 0, j \geq 2, i+j > n+1, \end{cases} \quad e_i f_1 = \begin{cases} (i^3 - i) f_{i+1}, 1 \leq i \leq n, \\ 0, \ i > n. \end{cases}; \tag{16}$$

- $\tilde{V}_{1,-n}(n+1) = \tilde{V}^*_{-2,-3}(n+1)$, $k = n$.

Remark 6. (1) There is another relation of duality

$$V^*_{\lambda,2\lambda-k}(n+1) = V_{-\lambda-1,-2\lambda-2-k}(n+1).$$

(2) $\tilde{V}_{0,-1}(n+1) = \tilde{V}_{-1,-2}(n+1)$, where $\tilde{V}_{-1,-2}(n+1)$ is defined by

$$e_if_j = \begin{cases} (j+i-1)f_{i+j}, j \geq 2, i+j \leq n+1, \\ 0, j \geq 2, i+j > n+1, \end{cases} \quad e_if_1 = \begin{cases} i(i-1)f_{i+1}, 1 \leq i \leq n, \\ 0, \, i > n. \end{cases} \tag{17}$$

Undeformed modules are non-isomorphic $V_{0,-1}(n+1) \neq V_{-1,-2}(n+1)$. The module $V_{0,-1}(n+1)$ is decomposable $V_{0,-1}(n+1) = \langle f_1 \rangle \oplus \langle f_2, \dots, f_{n+1} \rangle$ and $V_{-1,-2}(n+1)$ is not. However their n-dimensional submodules $\langle f_2, \dots, f_{n+1} \rangle$ are isomorphic.

Proof. The first example $V_{\lambda,2\lambda-k}(n+1)$ in the list of graded thread W^+-modules from the Theorem is absolutely ovbious, we have for all $j, 1 \leq j \leq n$

$$e_1f_j = (j + 2\lambda - k - 2\lambda)f_{j+1} = (j-k)f_{j+1}.$$

What other graded thread W^+-modules exist of the type $(1, \dots, 1, 0, 1, \dots, 1)$?

Lemma 4. *Let V be a $(n+1)$-dimensional graded thread W^+-module of the type $(1, \dots, 1, 0, 1, \dots, 1)$ with $\alpha_k = 0, 1 \leq k \leq n$, i.e. $e_1f_k = 0$. The corresponding graded W^+-module V is decomposable in a direct sum of two graded W^+-modules:*

$$V = V_1 \oplus V_2, \;\; V_1 = \langle f_1, \dots, f_k \rangle, V_2 = \langle f_{k+1}, \dots, f_{n+1} \rangle.$$

if and only if $\beta_k = \beta_{k-1} = 0$, i.e. $e_2f_{k-1} = 0, e_2f_k = 0$. We denoted by $f_1, \dots, f_{n+1}$ the graded basis of V.

Proof. It is evident that the subspace V_2 is invariant. On the another hand, $e_1f_k = e_2f_k = e_2f_{k-1} = 0$. Hence, the subspace V_1 is invariant with respect to e_1, e_2 and therefore it is invariant with respect to the entire W^+-action. □

Now, we have to rewrite the basic equations (3) for a module with $\alpha_k = 0$. It is easy to see that one have to substitute b_i by $\frac{\beta_i}{\alpha_i\alpha_{i+1}}$ in (3) and then multiply R^5_i by the product $\alpha_i\alpha_{i+1}\alpha_{i+2}\alpha_{i+3}$ and R^7_i by $\alpha_i\alpha_{i+1}\dots\alpha_{i+5}$ respectively. We suppose that α_i=1, $i \neq k$, α_k=0. One can meet α_k only in the denominators of b_k and b_{k-1}. Hence, the new equation that involves b_k is obtained from the old one by a very simple procedure: we keep summands only of the form b_kb_j or $b_{k-1}b_l$, for instance if $1 < k < n-2$ we have

$$R^5_{k-1}: \quad b_{k+2}(b_{k-1}-b_k)-b_{k-1}(b_{k+1}-b_{k+2}) = b_{k-1}-3b_k,$$

$$R^7_{k-1}: \quad b_{k+4}(b_{k-1}-3b_k)-b_{k-1}(b_{k+1}-3b_{k+2}+3b_{k+3}-b_{k+4}) = \frac{9}{10}(b_{k-1}-5b_k). \tag{18}$$

It follows from Lemma 4 that for an indecomposable module V with $\alpha_k = 0$ the constants b_k, b_{k-1} can not vanish simultaneously.

Lemma 5. *Let V be a 9-dimensional graded thread W^+-module defined by its basis $f_k, f_{k+1}, \dots, f_{k+8}$, and the defining set of relations:*

$$\tilde{e}_1 f_k = 0,\ \tilde{e}_1 f_i = f_{i+1},\ i = k+1, \dots, k+7;$$
$$\tilde{e}_2 f_j = b_j f_{j+2}, \quad j = k, \dots, k+6.$$

Then, V is either decomposable as a direct sum of W^+-modules $\langle f_k \rangle \oplus \langle f_{k+1}, \dots, f_{k+8} \rangle$ ($b_k = 0$) or it is indecomposable ($b_k \neq 0$) and V isomorphic to the one and only one graded thread W^+-module with the defining set $(b_k, \dots, b_{k+6})$ from the table below

module	b_k	b_{k+1}	b_{k+2}	b_{k+3}	b_{k+4}	b_{k+5}	b_{k+6}
1 $u \neq 1$	$*$	$\frac{6(u+1)}{1\cdot 2}$	$\frac{6(u+2)}{2\cdot 3}$	$\frac{6(u+3)}{3\cdot 4}$	$\frac{6(u+4)}{4\cdot 5}$	$\frac{6(u+5)}{5\cdot 6}$	$\frac{6(u+6)}{6\cdot 7}$
2	$*$	t	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$
3	$*$	$\frac{5}{2}$	$\frac{9}{5}$	$\frac{7}{5}$	$\frac{8}{7}$	$\frac{27}{28}$	$\frac{5}{6}$
4	$*$	1	$\frac{9}{5}$	$\frac{7}{5}$	1	$\frac{3}{4}$	$\frac{17}{28}$

(19)

Proof. Consider two equations with b_k.

$$\begin{aligned} R_k^5: &\quad b_k(2b_{k+3} - b_{k+2} - 1) = 0, \\ R_k^7: &\quad b_k\left(2b_{k+5} - b_{k+2} + 3b_{k+3} - 3b_{k+4} - \frac{9}{10}\right) = 0. \end{aligned} \tag{20}$$

Remark that if $b_k = 0$ then for the first basis vector f_k we have

$$e_1 f_k = 0, e_2 f_k = 0,$$

and hence the one-dimensional subspace $\langle f_k \rangle$ is invariant with respect to the entire W^+-action.

If $b_k \neq 0$ we have two linear equations (20) on $b_{k+2}, b_{k+3}, b_{k+4}, b_{k+5}$.

Now, we can apply the description of 8-dimensional graded thread modules from the Theorem 3. We consider the submodule $\langle f_{k+1}, \dots, f_{k+8} \rangle$ of V as a 8-dimensional graded thread module of the type $(1, 1, \dots, 1)$.

(1) Coordinates $b_{k+i} = \frac{6(u+i)}{(v+i)(v+i+1)}, i = 1, \dots, 6$, of a point $P(u, v) \in M_1$ satisfy both equations R_k^5 and R_k^7 if and only if

- $u{=}4, v{=}2$, i.e. $b_{k+1}{=}\frac{5}{2}, b_{k+2}{=}\frac{9}{5}, b_{k+3}{=}\frac{7}{5}, b_{k+4}{=}\frac{8}{7}, b_{k+5}{=}\frac{27}{28}, b_{k+6}{=}\frac{5}{6}$;
- $v{=}0$, i.e. $b_{k+i}{=}\frac{6(u+i)}{i(i+1)},\ i = 1, \dots, 6,\ u \neq 0, 1$.

(2) Coordinates $b_{k+i}{=}\frac{6}{(v+i)}$ of a point $P(v) \in M_1^0$ satisfy R_k^5 and R_k^7 also in two cases

- $v = 0$, i.e. $b_{k+1}{=}6, b_{k+2}{=}3, b_{k+3}{=}2, b_{k+4}{=}\frac{3}{2}, b_{k+5}{=}\frac{6}{5}, b_{k+6}{=}1$;
- $v = 1$, i.e. $b_{k+1}{=}3, b_{k+2}{=}2, b_{k+3}{=}\frac{3}{2}, b_{k+4}{=}\frac{6}{5}, b_{k+5}{=}1, b_{k+6}{=}\frac{6}{7}$.

Hence, we have to remove the restriction $u \neq 0, 1$ in the first line of our table.

(3) There are only two points in M_2 satisfying R_k^5 and R_k^7:

- $x=\frac{5}{2}, y=\frac{7}{4}$, i.e. $b_{k+1}=\frac{9}{2}, b_{k+2}=\frac{5}{2}, b_{k+3}=\frac{7}{4}, b_{k+4}=\frac{27}{20}, b_{k+5}=\frac{11}{20}, b_{k+6}=\frac{13}{14}$;
- $x=\frac{9}{5}, y=\frac{7}{5}$, i.e. $b_{k+1}=1, b_{k+2}=\frac{9}{5}, b_{k+3}=\frac{7}{5}, b_{k+4}=1, b_{k+5}=\frac{3}{4}, b_{k+6}=\frac{17}{28}$.

But the point with parameters $x=\frac{5}{2}, y=\frac{7}{4}$ coincides with the point $P(\frac{1}{2},0) \in M_1$ with $u=\frac{1}{2}, v=0$ that we have already considered above.

(4) Coordinates $(b_{k+1}, b_{k+2}, \dots, b_{k+6})$ of a point $P(t) \in M_3$ satisfies R^5_k only if $t=1$. However $P(1)$ does not satisfy R^7_k.

(5) It is easy to verify directly that there is no point in the subsets $M_4^{\pm}$, $M_5^{\pm}$,M_6^- that satisfies the equation R^5_k.

(6) All points from M_6^+ satisfy both equations (20). However it does not hold for points from M_6^-. □

Corollary 3. *Let V be a idecomposable $(n+1)$-dimensional graded thread W^+-module defined by its basis $f_1, f_2, \dots, f_{n+1}, n+1 \geq 10$ and the defining set of relations:*

$$\begin{aligned} e_1 f_1 = 0,\ e_1 f_i = f_{i+1},\ i=1,\dots,n;\\ e_2 f_j = b_j f_{j+2}, \quad j=1,\dots,n{-}1. \end{aligned}$$

then, V is isomorphic to the one and only one W^+-module from the table below

module	b_1	b_2	b_3	$\dots$	b_{i+1}	$\dots$	b_{n-1}
$V_{\lambda,-1+2\lambda}$, $\lambda \neq 0,-1$	-6λ	$\frac{6(-\lambda+1)}{1\cdot 2}$	$\frac{6(-\lambda+2)}{2\cdot 3}$	$\dots$	$\frac{6(-\lambda+i)}{i(i+1)}$	$\dots$	$\frac{6(-\lambda+n-2)}{(n-2)(n-1)}$
$\tilde{V}_{0,-1}$	6	$\frac{6}{2}$	$\frac{6}{3}$	$\dots$	$\frac{6}{i+1}$	$\dots$	$\frac{6}{n-1}$
$V^t_{-1,-3}$	6	t	3	$\dots$	$\frac{6}{i}$	$\dots$	$\frac{6}{n-2}$
$\tilde{V}_{-2,-3}$	6	$\frac{5}{2}$	$\frac{9}{5}$	$\dots$	$\frac{6(4+i)}{(2+i)(3+i)}$	$\dots$	$\frac{6(n+2)}{n(n+1)}$

Proof. We rescaled the first vector f_1 in order to fix the value of b_1 (we recall that b_1 is not determined by equations R^5_1 and R^7_1). It is possible because $e_1 f_1 = 0$.

We can conclude that if $b_k \neq 0$, then $b_{k+1}, \dots, b_{k+6}$ have the values prescribed by Lemma 20. It follows from Theorem 1 that $b_{k+i}, 1 \leq i \leq s$ if $7 \leq s \leq n-9$ is determined uniquely for all three subcases of (19)

$$(1)\ b_{k+i}=\frac{6(u+i)}{i(i+1)}; \quad 2)\ b_{k+i}=\frac{6}{i},\ ; \quad 3)\ b_{k+i}=\frac{6(i+4)}{(i+2)(i+3)}.$$

However, for the subcase

$$(4)\ \ b_k=1,\ b_{k+1}=1, b_{k+2}=\frac{9}{5},\ b_{k+3}=\frac{7}{5},\ b_{k+4}=1,\ b_{k+5}=\frac{3}{4},\ b_{k+6}=\frac{17}{28}.$$

the defining set can not be extended to a system $\{b_{k+1}, \dots, b_{k+7}\}$. It follows from the fact that 4) corresponds to the point $\left(1, \frac{9}{5}, \frac{7}{5}, 1, \frac{3}{4}, \frac{17}{28}\right)$ of the subset M_2 defined in Theorem 1 by parameters $x=y+\frac{2}{5}=\frac{9}{5}, y=\frac{7}{5}$. This set can not be extended to $\left(1, \frac{9}{5}, \frac{7}{5}, 1, \frac{3}{4}, \frac{17}{28}, b_{k+7}\right)$. One can also very it directly considering the equations R^5_{k+7}, R^7_{k+7}.

(1) For convenience of notations, we denote by $V^t_{-1,-3}$ a linear deformation of $V_{-1,-3}$ (t is a parameter). It is defined by the formulas

$$e_i f_j = (j+i-2) f_{i+j}, j \neq 2, i+j \leq n+1,$$
$$e_i f_2 = i(t(i-1) - i + 2) f_{i+2}, i \leq n-1.$$

(2) The W^+-module $\tilde{V}_{0,-1}$ is also deformed W^+-module $V_{0,-1}$. It is defined by

$$e_i f_j = (j+i+1) f_{i+j}, j \geq 2, i+j \leq n+1,$$
$$e_i f_1 = (i-1) f_{i+1}, 1 \leq i \leq n.$$

(3) The W^+-module $\tilde{V}_{-2,-3}$ is defined by

$$e_i f_j = (j+2i-1) f_{i+j}, j \geq 2, i+j \leq n+1,$$
$$e_i f_1 = (i^3 - i) f_{i+1}, 1 \leq i \leq n.$$

We also substituted for convinience $u = -\lambda$. □

Lemma 6. *Let V be a $k+8$-dimensional graded thread W^+-module defined by its basis $f_1, \ldots, f_k, f_{k+1}, \ldots, f_{k+8}, 2 \leq k \leq 8$, and the defining set of relations:*

$$e_1 f_k = 0,\ e_1 f_i = f_{i+1},\ i = 1, \ldots, k-1, k+1, \ldots, k+7;$$
$$e_2 f_j = b_j f_{j+2}, \quad j = 1, \ldots, k+6.$$

Then, V is either decomposable as a direct sum of W^+-modules $\langle f_1, \ldots, f_k \rangle \oplus \langle f_{k+1}, \ldots, f_{k+8} \rangle$ ($b_k = 0$) or it is indecomposable ($b_k \neq 0$) and V isomorphic to the one and only one graded thread W^+-module with the defining set $(b_1, \ldots, b_{k+6})$ from the table below

module	b_{k-i}	...	b_{k-2}	b_{k-1}	b_k	b_{k+1}	...	b_{k+i}	...	b_{k+6}
1	$\frac{6(u-i)}{i(i-1)}$	...	$\frac{6(u-2)}{1\cdot 2}$	$-6(u-1)$	$6u$	$\frac{6(u+1)}{1\cdot 2}$	...	$\frac{6(u+i)}{i(i+1)}$	...	$\frac{6(u+6)}{6\cdot 7}$
2	$-\frac{6}{i}$	...	$-\frac{6}{2}$	0	6	t	...	$\frac{6}{i}$	...	$\frac{6}{6}$
3	$-\frac{6}{i-1}$	...	t	-6	0	$\frac{6}{2}$	...	$\frac{6}{i+1}$	...	$\frac{6}{7}$

(21)

Proof. (1) Consider equations R^5_{k-1} and R^7_{k-1}

$$R^5_{k-1}: \quad b_{k-1}(2b_{k+2} - b_{k+1} - 1) = b_k(b_{k+2} - 3),$$
$$R^7_{k-1}: \quad b_{k-1}\left(2b_{k+4} - b_{k+1} + 3b_{k+2} - 3b_{k+3} - \frac{9}{10}\right) = 3b_k\left(b_{k+4} - \frac{3}{2}\right). \qquad (22)$$

In the subcase (1) from the Table (19) both equations are equivalent to

$$-b_{k-1}u = b_k(u-1).$$

Hence one can take $b_k = 6u\gamma,\ b_{k-1} = -6(u-1)\gamma,\ u \neq 0, 1, \gamma \neq 0$. If $u = 0$ then $b_k = 0$, if $u = 1$ then $b_{k-1} = 0$. After (if necessary) rescaling of f_k we may assume that $\gamma = 1$.

In the subcase (2) we have $b_k \neq 0, b_{k+1}{=}t, b_{k+2}{=}\frac{6}{2}, b_{k+3}{=}\frac{6}{3}, b_{k+4}{=}\frac{6}{4}$. Then R^5_{k-1} and R^7_{k-1} are

$$b_{k-1}\left(5-t\right)=0, \quad b_{k-1}\left(\frac{51}{10}-t\right)=0.$$

That implies $b_{k-1}=0$. We set $b_k=6$.

For the subcase (3) in (19) the system R^5_{k-1} and R^7_{k-1} look

$$b_{k-1}\frac{1}{10}=-b_k\frac{6}{5}, \quad b_{k-1}\frac{3}{35}=-b_k\frac{15}{14}$$

and it is inconsistent. Hence the case 3) is not extendable to the left.

The subcase (4) also leeds to a inconsistent system on unknowns b_{k-1} and b_k and also is not extendable.

Now, we have to study the case $b_k{=}0$. As $b_{k-1} \neq 0$ we remark that (22) is equivalent to the following linear system

$$\begin{aligned} 2b_{k+2}{-}b_{k+1}{-}1&=0,\\ 2b_{k+4}{-}b_{k+1}{+}3b_{k+2}{-}3b_{k+3}{-}\frac{9}{10}&=0. \end{aligned} \tag{23}$$

One can remark that the Eq. (23) can be obtained from (20) just by shifting the index $k \to k-1$. The mimic of the proof of the Lemma 5 will give us the following answer (b_{k-1} can take arbitrary values but after rescaling (if necessary) we may assume that $b_{k-1}{=}{-}6$).

$$b_k{=}0, b_{k+1}{=}\frac{6}{2}, b_{k+2}{=}\frac{6}{3}, b_{k+3}{=}\frac{6}{4}, b_{k+4}{=}\frac{6}{5}, b_{k+5}{=}\frac{6}{6}, b_{k+6}{=}\frac{6}{7}.$$

We summarize our results by means of the following table:

module	b_{k-1}	b_k	b_{k+1}	b_{k+2}	b_{k+3}	b_{k+4}	b_{k+5}	b_{k+6}
1	$-6(u{-}1)$	$6u$	$\frac{6(u+1)}{1\cdot 2}$	$\frac{6(u+2)}{2\cdot 3}$	$\frac{6(u+3)}{3\cdot 4}$	$\frac{6(u+4)}{4\cdot 5}$	$\frac{6(u+5)}{5\cdot 6}$	$\frac{6(u+6)}{6\cdot 7}$
2	0	6	t	3	2	$\frac{3}{2}$	$\frac{6}{5}$	1
3	-6	0	3	2	$\frac{3}{2}$	$\frac{6}{5}$	1	$\frac{6}{7}$

(24)

(2) Suppose, now, that $k \geq 3$. Hence, we may consider equations R^5_{k-2} and R^7_{k-2}

$$\begin{aligned} R^5_{k-2}: \quad b_{k-2}b_k &= 3b_{k-1}{-}3b_k{-}b_{k-1}b_{k+1},\\ R^7_{k-2}: \quad b_{k-2}b_k &= \frac{9}{2}b_{k-1}{-}9b_k{-}3b_{k-1}b_{k+3}{+}3b_kb_{k+3}. \end{aligned} \tag{25}$$

For the module (1) in (24) the Eq. (25) are equivalent to one equation

$$b_{k-2}u=3u(u{-}2).$$

Hence, we take $b_{k-2}=3(u{-}2)=\frac{6(u-2)}{1\cdot 2}$ if $u \neq 0$.

For the module (2) in (24) our equations imply $b_{k-2}=-3$.

In the subcase (3) taking into account b_k=0 we conclude that all values of b_{k-2} are valid and we set $b_{k-2} = t$.

(3) Suppose, now, that $k \geq 4$. The equations on b_{k-3} are the following ones

$$\begin{aligned} R^5_{k-3}: \quad b_{k-3}(2b_k - b_{k-1}) &= b_{k-2}b_k + 3b_{k-1} - b_k, \\ R^7_{k-3}: \quad b_{k-3}(b_{k-1} - 3b_k) &= b_{k+2}(3b_{k-1} - b_k) - 9b_{k-1} + 9b_k, \end{aligned} \tag{26}$$

For the first module (1) they are equivalent to

$$b_{k-3}(3u-1) = (3u-1)(u-3), \quad b_{k-3}(-4u+1) = (-4u+1)(u-3).$$

Hence, $b_{k-3} = u-3$. For the second module (2) both equations are equivalent to $b_{k-3} = -2$. For the module (3) we have $b_{k-3} = -3$.

(4) Let $k \geq 5$. The equations on b_{k-4} are

$$\begin{aligned} R^5_{k-4}: \quad b_{k-1}(2b_{k-4} - 3b_{k-3} + 1) &= 0, \\ R^7_{k-4}: \quad b_{k-3}(3b_{k-1} - 3b_k) &= b_{k-1}b_{k+1} + \frac{9}{2}b_k - 9b_{k-1}, \end{aligned} \tag{27}$$

they will give us for (1), (2), (3) respectively

$$(1)\ b_{k-4} = \frac{u-4}{2}, \quad (2)\ b_{k-4} = -\frac{3}{2}, \quad (3)\ b_{k-4} = -2.$$

(5) The remaining cases ($k \geq 6, 7$) are treated similarly to the previous ones. $\square$

Now considering a general case of $(n+1)$-dimensional graded thread W^+-module V, $n+1 \geq 16$, with $e_1 f_k = 0$ we may assume that $k+8 \leq n$. If $k > n-8 \geq 7$ we take the dual W^+-module V^* instead of V. We have for V^*

$$(\alpha^*_1, \alpha^*_2, \ldots, \alpha^*_{n-1}, \alpha^*_n) = (-\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_2, -\alpha_1).$$

It means that $\alpha^*_{n+1-k} = -\alpha_k = 0$ and $(n+1-k)+7 \leq 15 \leq n$.

Assuming $k+8 \leq n$, we can define a 9-dimensional subquotient $\tilde{V}$ of V

$$\tilde{V} = \langle f_k, \ldots, f_{k+7}, f_{k+8}, \ldots, f_n \rangle / \langle f_{k+8}, \ldots, f_n \rangle$$

and apply Lemmas 5, 6 and Corollary 3.

Every graded thread W^+-module $\tilde{V}$ (the set $(b_k, \ldots, b_{k+7})$) presented in the classification lists of Lemmas 5, 6 and Corollary 3 can be uniquely extended to the graded thread W^+-module V $((b_1, \ldots, b_k, \ldots, b_{k+8}, \ldots, b_{n-1})$. We remove the restriction $k+8 \leq n$ considering their dual modules.

The results of this classification are presented in the Table 2. $\square$

Table 2. Graded thread W^+-modules of the type $(1,\dots,1,0,1\dots,1), \dim V = n+1 \geq 16$.

Module	b_1	b_2	$\dots$	b_{k-2}	b_{k-1}	b_k	b_{k+1}	$\dots$	b_{n-1}
$V_{\lambda,2\lambda-k}(n+1)$, $\lambda \neq 0,-1$, $2<k<n-1$	$\frac{6(-\lambda-k+1)}{(k-1)(k-2)}$	$\frac{6(-\lambda-k+2)}{(k-2)(k-3)}$	$\dots$	$\frac{6(-\lambda-2)}{2\cdot 1}$	$6(-\lambda-1)$	-6λ	$\frac{6(-\lambda+1)}{1\cdot 2}$	$\dots$	$\frac{6(-\lambda+n-1-k)}{(n-k-1)(n-k)}$
$V_{\lambda,2\lambda-2}(n+1)$, $\lambda \neq -1, k=2$	$6(\lambda+1)$	-6λ	$\dots$	$\dots$	$\frac{6(-\lambda+k-3)}{(k-3)(k-2)}$	$\dots$	$\frac{6(-\lambda+k-1)}{(k-1)k}$	$\dots$	$\frac{6(-\lambda+n-3)}{(n-3)(n-2)}$
$V_{\lambda,2\lambda-1}(n+1)$, $\lambda \neq 0,-1, k=1$	-6λ	$\frac{6(-\lambda+1)}{1\cdot 2}$	$\dots$	$\dots$	$\frac{6(-\lambda+k-2)}{(k-2)(k-1)}$	$\dots$	$\frac{6(-\lambda+k)}{k(k+1)}$	$\dots$	$\frac{6(-\lambda+n-2)}{(n-2)(n-1)}$
$V_{\lambda,2\lambda-n}(n+1)$, $\lambda \neq 0,-1, k=n$	$\frac{6(-\lambda-n+1)}{(n-2)(n-1)}$	$\frac{6(-\lambda-n+2)}{(n-2)(n-3)}$	$\dots$	$\dots$	$\frac{6(-\lambda-n+k-1)}{(n-k+2)(n-k+1)}$	$\dots$	$\frac{6(-\lambda-n+k)}{(n-k)(n-k-1)}$	$\dots$	$6(-\lambda-1)$
$V_{\lambda,2\lambda-n+1}(n+1)$, $\lambda \neq 0, k=n-1$	$\frac{6(-\lambda-n+2)}{(n-2)(n-3)}$	$\frac{6(-\lambda-n+3)}{(n-3)(n-4)}$	$\dots$	$\dots$	$\frac{6(-\lambda-n+k)}{(n-k)(n-k-1)}$	$\dots$	$\frac{6(-\lambda-n+k+2)}{(n-k-2)(n-k-3)}$	$\dots$	-6λ
$V^t_{-1,-k-2}(n+1)$, $2<k<n-1$	$-\frac{6}{k-1}$	$-\frac{6}{k-2}$	$\dots$	$-\frac{6}{2}$	0	6	t	$\dots$	$\frac{6}{n-k-1}$
$V^t_{-1,-4}(n+1)$, $k=2$	0	6	$\dots$	$\frac{6}{k-4}$	$\frac{6}{k-3}$	$\frac{6}{k-2}$	$\frac{6}{k-1}$	$\dots$	$\frac{6}{n-3}$
$V^t_{-1,-3}(n+1)$, $k=1$	6	t	$\dots$	$\frac{6}{k-3}$	$\frac{6}{k-2}$	$\frac{6}{k-1}$	$\frac{6}{k}$	$\dots$	$\frac{6}{n-2}$
$V^t_{0,-k}(n+1)$, $2<k<n-1$	$-\frac{6}{(k-2)}$	$-\frac{6}{(k-3)}$	$\dots$	t	-6	0	$\frac{6}{2}$	$\dots$	$\frac{6}{n-k}$
$V^t_{0,-n+1}(n+1)$, $k=n-1$	$-\frac{6}{n-3}$	$-\frac{6}{n-2}$	$\dots$	$-\frac{6}{n-k}$	$-\frac{6}{n-k-1}$	$-\frac{6}{n-k-2}$	$-\frac{6}{n-k-3}$	$\dots$	0
$V^t_{0,-n}(n+1)$, $k=n$	$-\frac{6}{n-2}$	$-\frac{6}{n-1}$	$\dots$	$-\frac{6}{n-k+1}$	$-\frac{6}{n-k}$	$-\frac{6}{n-k-1}$	$-\frac{6}{n-k-2}$	$\dots$	-6
$\tilde{V}_{0,-1}(n+1)$, $k=1$	6	$\frac{6}{2}$	$\dots$	$\frac{6}{k-2}$	$\frac{6}{k-1}$	$\frac{6}{k}$	$\frac{6}{k+1}$	$\dots$	$\frac{6}{n-1}$
$\tilde{V}_{-1,-2-n}(n+1)$, $k=n$	$-\frac{6}{n-1}$	$-\frac{6}{n-2}$	$\dots$	$-\frac{6}{n-k+2}$	$-\frac{6}{n-k+1}$	$\frac{6}{n-k}$	$\frac{6}{n-k-1}$	$\dots$	-6
$\tilde{V}_{-2,-3}(n+1)$, $k=1$	6	$\frac{5}{2}$	$\dots$	$\frac{6(1+k)}{(k-1)k}$	$\frac{6(2+k)}{k(k+1)}$	$\frac{6(3+k)}{(k+1)(k+2)}$	$\frac{6(4+k)}{(k+2)(k+3)}$	$\dots$	$\frac{6(n+2)}{n(n+1)}$
$\tilde{V}_{1,-n}(n+1)$, $k=n$	$-\frac{6(n+2)}{n(n+1)}$	$-\frac{6(n+1)}{(n-1)n}$	$\dots$	$\dots$	$-\frac{6(n+4-k)}{(n+3-k)(n+2-k)}$	$\dots$	$-\frac{6(n+2-k)}{(n+1-k)(n-k)}$	$\dots$	-6

5 Graded Thread Modules of the Type $(1, \ldots, 1, 0, 0, 1, \ldots, 1)$.

Now, we consider modules with vanishing two consecutive α_i, α_{i+1}, i.e.

$$\exists!\ k, 1 \le k \le n-1,\ \alpha_k = \alpha_{k+1} = 0. \tag{28}$$

Modules of this type are called "reprisé" in [3]

In [12] an infinite-dimensional graded thread W^+-module $\tilde{V}_{gr}$ was constructed, it was defined by its basis $\{f_j, j \in \mathbb{Z}\}$ and the relations

$$e_i f_j = \begin{cases} j f_{i+j}, & j \ge 0; \\ (i+j) f_{i+j}, & i+j \le 0,\, j < 0; \\ f_{i+j}, & i+j > 0,\, j < 0. \end{cases} \tag{29}$$

It holds for this module $e_1 f_{-1} = e_1 f_0 = 0$. This module and its finite-dimensional subquotients played the crucial role in the proof of Buchstaber's conjecture on Massey products in Lie algebra cohomology $H^*(W^+, \mathbb{K})$ [12]. It has interesting nature, it is not a module of $V_{\lambda,\mu}$ family or its degeneration or deformation, in some sense it is the result of gluing together of two modules: the quotient of $V_{-1,1}$ with a submodule of $V_{0,0}$ and it is unique infinite-dimensional module with the property $\exists!\ k, \alpha_k = \alpha_{k+1} = 0$ [12] .

Theorem 3. *Let V be a $(n+1)$-dimensional, $n+1 \ge 11$, indecomposable graded thread W^+-module of the type $(1, \ldots, 1, 0, 0, 1, \ldots, 1)$, i.e. there exists a basis $f_1, \ldots, f_{n+1}$ of V and $\exists! k, 1 \le k \le n-1$ such that:*

$$\tilde{e}_1 f_i = f_{i+1},\ i = 1, \ldots, k-1, k+2, \ldots, n-1;$$
$$\tilde{e}_1 f_k = e_1 f_{k+1} = 0, \quad \tilde{e}_2 f_j = b_j f_{j+2}, \quad j = 1, \ldots, n-2.$$

then if $k \ne 1, n-1$, the module V is isomorphic to one and only one module from the list

- *$R_k, 1 \le k \le n-1$, defined by its basis $f_1, \ldots, f_{n+1}$ and relations*

$$e_i f_j = \begin{cases} (j-k-1) f_{i+j}, & k+1 \le j \le n+1,\ i+j \le n+1; \\ (i+j-k-1) f_{i+j}, & i+j \le k+1, j < k+1; \\ f_{i+j}, & k+1 < i+j \le n+1, j < k+1; \\ 0, & \text{otherwise}. \end{cases} \tag{30}$$

- *its dual module $R_k^*, 1 \le k \le n-1$.*

Proof. The equations that involves b_k, b_{k+1}, b_{k+2} will be obtained from the standard ones by a very simple procedure: we will keep the summands only of the form $b_k b_j$, $b_{k+1} b_l$, $b_{k+2} b_l$.

(1) Let consider the case $k=1$ (and hence the dual module with $k=n-2$). The first four equations $R^5_1, R^5_2, R^5_3, R^7_1$ are:

$$\begin{aligned} 2b_1b_4 - b_1b_3 - b_1 &= 0, \\ 2b_2b_5 - b_2b_4 - b_2 &= 0, \\ 2b_3b_6 - b_4b_6 - b_3b_5 - (b_3 - 3b_4 + 3b_5 - b_6) &= 0, \\ b_6b_1 - b_1(b_3 - 3b_4 + 3b_5 - b_6) - \frac{9}{10}b_1 &= 0. \end{aligned} \tag{31}$$

If $b_1=0$, then there is a decomposition $V = \langle f_1\rangle \oplus \langle f_2, \dots, f_n\rangle$ in the sum of two submodules, if $b_2=0$ then $V = \langle f_2\rangle \oplus \langle f_1, f_3, \dots, f_n\rangle$ is also the sum of its submodules. Hence we may assume that $b_1=b_2=1$.

The system (31) with $b_1=b_2=1$ has two solutions

$$(b_3, b_4, b_5, b_6) = \left(3, 2, \frac{3}{2}, \frac{6}{5}\right), \left(\frac{9}{5}, \frac{7}{5}, \frac{6}{5}, \frac{21}{20}\right).$$

It follows from Proposition 5 that the module $\langle f_2, f_3, \dots, f_7\rangle$ corresponding to the second solution can not be extended to $\langle f_2, f_3, \dots, f_7, \dots, f_n\rangle$ with $n \geq 10$. On the another hand, one can check out that we have the only one module with

$$b_1 = 1, b_2 = 1, b_3 = 3, b_4 = 2, \dots, b_{n-2} = \frac{6}{n-3}$$

that corresponds to the first solution.

(2) Let us suppose, now, that $2 \leq k \leq n-6$. Then the equations $R^5_{k-1}, R^5_k, R^5_{k+1}, R^7_{k-1}$ will have the following form:

$$\begin{aligned} -b_kb_{k+2} - b_{k-1}b_{k+1} + 3b_k &= 0, \\ 2b_kb_{k+3} - b_kb_{k+2} - b_k &= 0, \\ 2b_{k+1}b_{k+4} - b_{k+1}b_{k+3} - b_{k+1} &= 0, \\ -3b_kb_{k+4} - b_{k-1}b_{k+1} + \frac{9}{2}b_k &= 0. \end{aligned} \tag{32}$$

Proposition 8. *Let $b_k=0$ or $b_{k-1}=b_{k+1}=0$ then the module V is decomposable.*

Proof. Indeed, it follows from the first equation of the system above that if $b_k=0$ then $b_{k-1}b_{k+1}=0$. In the case of $b_{k-1}=b_k=0$ we have the invariant decomposition:

$$V = \langle f_1, \dots, f_k\rangle \oplus \langle f_{k+1}, \dots, f_n\rangle.$$

If $b_k=b_{k+1}=0$ then V is decomposed in another way:

$$V = \langle f_1, \dots, f_k, f_{k+1}\rangle \oplus \langle f_{k+2}, \dots, f_n\rangle.$$

If both $b_{k-1}=b_{k+1}=0$ then V is also decomposable:

$$V = \langle f_1, \dots, f_k, f_{k+2}, \dots, f_n\rangle \oplus \langle f_{k+1}\rangle.$$

Now, after some rescaling of the basic vectors we have to consider two possibilities:

(1) $b_{k-1} = b_k = 1$;
(2) $b_{k-1} = 0, \ b_k = b_{k+1} = 1$.

In the first case, the system (32) has the only one solution

$$b_{k+1}{=}0, b_{k+2}{=}3, b_{k+3}{=}2, b_{k+4}{=}\frac{3}{2}.$$

In the second case, we also have the unique solution $b_{k+2}{=}3, b_{k+3}{=}2, b_{k+4}{=}\frac{3}{2}$. The components $b_i, i > k{+}4$ are also determined uniquely as it follows from the Proposition 5. If we want to find b_{k-2} we have to suppose that $k \geq 3$ and to consider two new equations:

$$\begin{aligned} b_{k-1}b_{k+1}{-}b_{k-2}b_k{-}3b_k &= 0,\\ 3b_kb_{k+3}{-}b_{k-2}b_k{-}9b_k &= 0. \end{aligned} \tag{33}$$

Evidently in both cases ($b_{k+1} = 0$ or $b_{k+1} = 1$) we have the same answer $b_{k-2} = -3$.

Now, supposing that $k \geq 4$ we have two new additional equations:

$$\begin{aligned} 2b_{k-3}b_k{-}b_{k-2}b_k{+}b_k &= 0,\\ -b_kb_{k+2}{+}3b_{k-3}b_k{+}9b_k &= 0. \end{aligned} \tag{34}$$

Again it follows that $b_{k-3} = -2$ in both situations.

Let $k \geq 5$. We have two equations on b_{k-4}:

$$\begin{aligned} 2b_{k-4}b_{k-1}{-}b_{k-3}b_{k-1}{+}b_{k-1} &= 0,\\ -b_{k+1}b_{k-1}{-}3b_{k-4}b_k{-}\frac{9}{2}b_k &= 0. \end{aligned} \tag{35}$$

In both cases we have $b_{k-4} = -\frac{3}{2}$.

The last case $k \geq 6$ can be treated absolutely similarly. In fact we have obtained two graded modules:

module	b_1	...	b_{k-2}	b_{k-1}	b_k	b_{k+1}	b_{k+2}	...	b_{n-1}
R_k	$-\frac{6}{k-1}$	...	-3	1	1	0	3	...	$\frac{6}{n-k-1}$
R^*_{n-k-1}	$-\frac{6}{k-1}$	...	-3	0	1	1	3	...	$\frac{6}{n-k-1}$

The cases $n{-}5 \leq k \leq n{-}3$ follow from the previous considerations: one have to take the corresponding dual module instead of initial one. $\square$

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Localized Solutions of the Schrödinger Equation on Hybrid Spaces. Relation to the Behavior of Geodesics and to Analytic Number Theory

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Abstract. In this chapter, we review our results concerning localized asymptotic solutions of time-dependent Schrödinger equation on hybrid spaces. We describe the connections of this problem to the problem of global behavior of geodesics on Riemannian manifolds and to certain problems of the analytic number theory.

Keywords: Schrödinger equation · Hybrid spaces
Localised asymptotic solutions of time-dependent Schrödinger equation

1 Introduction

Differential operators on hybrid spaces—topological spaces of variable dimensions, obtained by connecting a number of Riemannian manifolds by arcs of curves—are intensively studied during last decades (see, e.g., [1–5] and references therein). The corresponding problems have various applications (quantum theory of complex molecules, waves in thin structures, traffic problems, neural impulses etc.). In the papers [5,6] we discussed the Cauchy problem for time-dependent Schrödinger equation on such a space. Namely, we studied propagation of localized initial state (Gaussian packet); certain properties of the corresponding semi-classical asymptotics appeared to be close to the properties of an analogous problem on a metric graph. In particular, there exists a close connection between statistics of localized solutions and certain problems of analytic number theory (for metric graphs, we studied such connections previously, see, e.g. [7,8]). However, for hybrid spaces, the connections are more wide and depend on the properties of geodesic flows on the corresponding Riemannian manifolds. The nature of this correspondence is purely classical—it can be described in terms of geodesic curves and geodesic spheres. Here we review our results on the behavior of localized wave packets (quasi-particles) and discuss the connections mentioned above. The proofs can be found in [5,7–9]. The proofs of the last two statements will be published in a separate paper.

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1.1 Hybrid Spaces and Schrödinger Operators

Consider a finite number of geodesically complete smooth Riemannian manifolds $M_1, \dots, M_k$ $\dim M_r \le 3$ and a finite number of segments $\gamma_1, \dots, \gamma_s$; the segments are also endowed with metrics.

Definition 1. The *hybrid space* Γ is a topological space, obtained by identifying the endpoints of the segments with certain points on the manifolds; we assume that different endpoints of the segments are glued to the different points of the manifolds. We will denote the points of gluing by q_j, $j = 1, \dots, 2s$.

Remark 1. In certain papers (see, e.g., [4]) hybrid spaces are called *decorated graphs.*

Remark 2. Replacing all manifolds by points, we obtain from Γ the finite metric graph; the results of the chapter are valid for such graphs also.

Further we will always assume that Γ is connected.

Let Q be an arbitrary real valued continuous function on Γ, smooth on the edges. Let Q_j and Q_r be restrictions of Q to γ_j and to M_r respectively. Consider a direct sum $\widehat{H}_0 = \bigoplus\limits_{j=1}^{s} \left(-\frac{h^2}{2}\frac{d^2}{dz_j^2} + Q_j\right) \bigoplus\limits_{r=1}^{k} \left(-\frac{h^2}{2}\Delta_r + Q_r\right)$ with the domain $H^2(\Gamma) = \bigoplus\limits_{j=1}^{s} H^2(\gamma_j) \bigoplus\limits_{r=1}^{k} H^2(M_r)$. Here $\frac{d^2}{dz_j^2}$ is an operator of the second derivative on γ_j with respect to a fixed parametrization with Neumann boundary conditions, Δ_r is the Laplace–Beltrami operator on M_r.

Definition 2. The *Schrödinger operator* $\widehat{H}$ is a self-adjoint extension of the restriction $\widehat{H_0}|_L$, where $L = \{\psi \in H^2(\Gamma), \quad \psi(q_j) = 0\}$.

Domain of the operator $\widehat{H}$ contains functions with singularities in the points q_j. Namely, let $G(x, q, \lambda)$ be the Green function on M_k (integral kernel of the resolvent) of Δ, corresponding to the spectral parameter λ. This function has the following asymptotics as $x \to q$: $G(x, q, \lambda) = F_0(x, q) + F_1$, where F_1 is a continuous function and F_0 is independent of λ and has the form

$$F_0 = -\frac{c_2}{2\pi}\ln\rho, \quad \dim M = 2; \tag{1}$$

$$F_0 = \frac{c_3}{4\pi\rho}, \quad \dim M = 3. \tag{2}$$

Here, $c_j(x, q)$ are continuous functions, $c_j(q, q) = 1$, ρ is the geodesic distance between x and q. The function ψ from the domain of the operator $\widehat{H}$ has the following asymptotics as $x \to q_j$: $\psi = \alpha_j F_0(x) + b_j + o(1)$, $\alpha_j, b_j \in \mathbf{C}$. Now, for each endpoint of the segment (i.e. for each point q) consider a pair $\psi(q)$, $h\psi'(q)$ and a vector $\xi = (u, v)$, $u = (h\psi'(q_1), \dots, h\psi'(q_{2s}), \alpha_1, \dots, \alpha_{2s})$, $v = (\psi(q_1), \dots, \psi(q_{2s}), hb_1, \dots, hb_{2s})$. Consider a standard skew-Hermitian form $[\xi^1, \xi^2] = \sum_{j=1}^{4s}(u_j^1\bar{v}_j^2 - v_j^1\bar{u}_j^2)$ in $\mathbf{C}^{4s} \oplus \mathbf{C}^{4s}$. Let us fix the Lagrangian plane

$\Lambda \subset \mathbf{C}^{4s} \oplus \mathbf{C}^{4s}$. Arbitrary self-adjoint extension $\widehat{H}$ is defined by the coupling conditions $\xi \in \Lambda$ or equivalently

$$-i(I+U)u + (I-U)v = 0,$$

where U is a unitary matrix defining Λ and I is an identity matrix. Physically it is more natural to consider local coupling conditions, $\Lambda = \bigoplus_q \Lambda_q$, where $\Lambda_q \subset \mathbf{C}^4$ is defined for each point q separately. Further we will always suppose that the coupling conditions are local and the plane Λ is in general position (roughly speaking, the coupling conditions do not have to divide Γ in several independent parts).

A time-dependent Schrödinger equation on the graph Γ is an equation of the form

$$ih\frac{\partial \psi}{\partial t} = \widehat{H}\psi, \quad h > 0. \tag{3}$$

We choose initial conditions that have the form of a narrow packet localized near the point z_0, which lies on the j-th edge of Γ:

$$\psi(z,0) = h^{-1/4}K(z_j)\exp\left(\frac{iS_0(z_j)}{h}\right). \tag{4}$$

$$S_0(x) = w(z_j - z_0)^2 + p_0(z_j - z_0),$$

where $z_0 \in \gamma_j$, $p_0 \in \mathbf{R}$, and w is complex, with $\Im(w) > 0$, $K(z_j)$ is a cut off function supported on the edge γ_j, $K = 1$ in the vicinity of z_0. Factor $h^{-1/4}$ is introduced to ensure that the initial function $\psi(z_j, 0)$ is of order of unity in the $L^2(\Gamma)$-norm. Due to the positivity of the imaginary part of w the initial function is localized in a small neighborhood of z_0: $\psi(z_j, 0) = o(h^N)\forall N$ with $|z_j - z_0| \geq \delta > 0$ (δ is independent of h). We assume that $\frac{1}{2}{p_0}^2 + Q(z_0) > Q(z), \forall z \in \Gamma$, which guarantees that there are no turning points (see, e.g., [5,8]) on Γ. Note that presence of turning points leads to a change of the space Γ: one has to cut Γ by the turning points and consider the connected component of the cut space, which contains z_0. Moreover, in order to study the semiclassical asymptotics, it is sufficient to consider the case $Q = 0$ (according to the Maupertuis principle, this can be achieved by the variation of metrics on M_l, γ_j); we will also assume that $p_0 = 1$. Further we describe the asymptotics of the Cauchy problem (3)–(4), as $h \to 0$.

2 Dynamics of Generalized Gaussian Packets on Hybrid Spaces

Now let Γ be a hybrid space and consider the Cauchy problem (3)–(4). If the time t is sufficiently small, the asymptotics is described by the following well-known statement.

Theorem 1. *The solution of the Cauchy problem (3)–(4), for* $t \in [0,T]$ *(*T *is sufficiently small), is given by the following formula*

$$\psi = h^{-1/4}\varphi(z_j,t)e^{iS(z_j,t)/h} + O(\sqrt{h}), \tag{5}$$

with $S(z_j,t) = S^0(t) + P(t)(z_j - Z(t)) + W(t)(z_j - Z(t))^2$*, where* $P(t), Z(t)$ *are solutions of the Hamiltonian system*

$$\dot{z} = H_p, \quad \dot{p} = -H_z, \quad H = \frac{1}{2}|p|^2, \quad Z(0) = z_0, \quad P(0) = 1.$$

Functions $\varphi(z_j,t)$*,* $S^0(t)$ *and* $W_j(t)$ *are explicitly expressed in terms of the solutions of the Hamiltonian system,* $ImW_j(t) > 0$*.*

Remark 3. This well-known statement means, that initially the packet moves along the classical trajectory on the edge.

Now we describe what happens at the time of scattering.

2.1 Scattering on Manifold

Let Γ be a half-line, connected with a manifold M in a single point q. Let t_0 be the instant of scattering (i.e. the time, when the trajectory of the classical Hamiltonian system on the half-line reaches q). Consider the sphere in T^*_qM: $L_0 : |p| = 1$. Consider the flow g_t of the classical Hamiltonian system on M with the Hamiltonian $H = \frac{1}{2}|p|^2$ and let L_t be the shifted sphere L_0: $L_t = g_tL_0$.

Theorem 2. *For certain time interval* $t \in (t_0, t_0+\varepsilon)$*, the solution of the Cauchy problem (3)–(4) has the form*

$$\psi = A(t)e^{\frac{iS(z,t)}{h}}, \quad z \in R_+ + O(\sqrt{h}), \tag{6}$$

$$\psi = K_{L_t}[B(x,t)] + O(\sqrt{h}), \quad x \in M. \tag{7}$$

Here, $S(z,t)$ *has the same form as in the Theorem 1),* K_{L_t} *is the Maslov canonic operator on isotropic manifold* L_t *with complex germ ([10]), functions* A *and* B *can be expressed explicitly in terms of the coupling matrix* U*.*

Remark 4. As $h \to 0$, the support of the function ψ tends to $\pi(L_t)$, where $\pi : T^*M \to M$ is the natural projection. In general, position ψ is localized near the surface of codimension 1; we call the function $K_{L_t}[B]$ a *generalized Gaussian packet* near the hypersurface. The set $\pi(L_t)$ is called *the support of the generalized Gaussian packet.*

Remark 5. Let Γ be an arbitrary hybrid space. During some time (neighborhood of the instant t_0) the solution will have the same form as described in the previous theorem. After some time the support of the generalized Gaussian packet reaches some gluing point q (it can coincide or not coincide with the point of the first scattering). At that time the packet produces one packet, propagating along the segment, glued at the point q and another, propagating inside the manifold. Then one of these packets reaches certain point of gluing and produces next 2 packets etc. It is easy to see that for an arbitrary time t the number of packets localized on the segments of Γ (not on the manifolds) can be defined as follows. Consider the geodesic on γ_j, starting from z_0 with the fixed unit velocity. At some instant of time the geodesic meets one of the points of gluing q. At this instant consider *all* geodesics on the corresponding manifold, starting from this point with unit velocities as well as the geodesic on the initial segment, starting from q in the direction, opposite to the direction of the initial geodesic. At certain instants the geodesics meet points of gluing; we consider all geodesics (on manifolds and on segments), starting from all these points with unit velocities. Clearly, for arbitrary instant of time we will have a set of points on the segments, propagating along the geodesics, and certain sets on the manifolds (union of geodesic spheres), moving along the geodesics. We suppose that for arbitrary time t the number of points, appearing on the segments, is finite and denote this number by $N(t)$. This number coincides with the number of packets; further we study the behavior of $N(t)$ as $t \to \infty$.

Definition 3. The number $N(t)$ is called *the number of quasi-particles* on the edges of Γ.

3 Statistics of Quasi-Particles on a Hybrid Space

3.1 The Counting Function for Geodesics

The behavior of the function $N(t)$ depends essentially on the properties of the geodesic flow on Γ. Namely, for each pair (q_i, q_j) of the points of gluing, lying on the same manifold M_r, consider the number $m_{ij}(t)$ of different lengths of geodesics on M_r, connecting q_i and q_j and such that these lengths are at most t (let us remind, that we assume that this number is finite for arbitrary t). Note that the points q_i and q_j can coincide. Let us denote by $m(t)$ the sum of $m_{ij}(t)$ for all pairs of points, lying on the same manifolds. The asymptotics of $N(t)$ is defined by the asymptotics of $m(t)$; the latter is defined by the properties of the geodesic flows on the manifolds M_j. We will consider three different situations.

3.2 The Finite Number of Geodesic Lengths

Lengths in General Position The simplest situation takes place if the total number of times of geodesics is finite (such situation appears, for example, if all M_r are Euclidean or hyperbolic spaces or spheres). We denote by $L_1, \dots, L_p$ these lengths and by $l_1, \dots l_s$ the lengths of the segments.

Theorem 3. *Let the set $L_1, \dots, L_p, l_1, \dots, l_s$ be linearly independent over the field* $\mathbf{Q}$. *Then the number of quasi-particles $N(t)$ has the following asymptotics as $t \to \infty$*

$$N(t) = Ct^{p+s-1} + o(t^{p+s-1}), \quad C = \text{const.} \tag{8}$$

The constant C can be computed explicitly.

Theorem 4. *The constant C in the previous theorem has the form*

$$C = \frac{\sum_{j=1}^{s} l_s}{2^{2s-2}(p+s-1)! \prod_{j=1}^{s} l_j \prod_{i=1}^{p} L_i} \tag{9}$$

Moreover, for almost all lengths the asymptotics can be computed with higher accuracy.

Theorem 5. *For almost all sets $l_1, \dots, l_s, L_1, \dots, L_p$ the function $N(t)$ is "almost polynomial"*

$$N(t) = \sum_{j=1}^{p+s-1} C_j t^t + o(t). \tag{10}$$

Remark 6. The previous theorems are valid for the metric graphs also. In this case one has to put $p+s=E$—the number of edges and $2s=v$—the number of vertices. For example, for the star graph with three edges the number of quasi-particles equals

$$N(t) = \frac{1}{8}\frac{l_1+l_2+l_3}{l_1 l_2 l_3}t^2 + \frac{1}{2}(\frac{1}{l_1}+\frac{1}{l_2}+\frac{1}{l_3})t + o(t). \tag{11}$$

Remark 7. The main step in the proof of the previous theorems is the following: the problem of the computation of $N(t)$ can be reduced to the problem of computation of the number of lattice points in certain growing polyhedra. The latter problem was studied by many experts in analytic number theory, and we use their results for our calculation of N. For example, the proof of the Theorem 5 uses the results of M. Skriganov ([11]); the formula (11) follows from the classical paper of Hardy and Littlewood on the number of lattice points in a triangle.

Uniform Distribution Distribution of quasi-particles on the edges of Γ appears to be uniform. Namely, let us consider a segment Δ on arbitrary edge of the space Γ and let $N_\Delta(t)$ denote the number of quasi-particles, localized on Δ.

Theorem 6. *For almost all sets $l_1, \dots, l_s, L_1, \dots, L_p$*

$$\lim_{t\to\infty} \frac{N_\Delta(t)}{N(t)} = \frac{|\Delta|}{l}, \tag{12}$$

where $|\Delta|$ denotes the length of the segment Δ and $l = \sum_{j=1}^{s} l_j$ is the total length of the edges of Γ.

Commensurable Lengths If the lengths of geodesics and those of the graph edges are commensurable, the number of quasi-particles grows slower. Computer experiments lead to the following conjecture.

Conjecture 1. Let r denote the rank of the set $l_1, \ldots, l_s, L_1, \ldots, L_p$ over $\mathbf{Q}$. Then

$$N(t) = Ct^{r-1} + o(t^{r-1}).$$

The proof is unknown even for metric graphs; however, there is a number of partial results in certain simple situations.

Proposition 1. *Consider a star graph with three edges with lengths* $l_1 = nl_0, l_2 = ml_0, l_3$*, where* $n \in \mathbf{N}, m \in \mathbf{N}$ *(GCD(n,m) = 1), and* l_3 *is such that rank* $\{l_1, l_2, l_3\}$ *over* $\mathbf{Q}$ *equals 2. Then the number of quasi-particles asymptotically equals*

$$N(t) = \frac{t}{2}\left(\frac{m+n}{l_3} + \frac{1}{l_0}\right) + o(t).$$

The number of quasi-particles stops to grow in the case where all lengths are commensurable ($r = 1$). The final number of quasi-particles can be computed for metric graphs.

Proposition 2. *Let* Γ *be a metric graph and* $l_j = n_j l_0$*, where* $n_j \in \mathbf{N}$ *and* $GCD(n_1, \ldots, n_s) = 1$*. Then, at some time the number of quasi-particles will cease to grow and will be equal to*

$$N = 2\sum_{i=1}^{s} n_i,$$

if Γ *contains a cycle with the length, which is not divisible by* $2l_0$*, or*

$$N = \sum_{i=1}^{s} n_i$$

otherwise.

Relation to the Frobenius Numbers Let Γ be a metric graph and let $r = 1$. The natural question is: at what instant of time the number of quasi-particles stops to grow? In the simplest cases this instant can be computed via Frobenius number of the lengths of the graph edges. Recall that the Frobenius number $Fr(n_1, ..., n_k)$ for a given set of positive integers $n_1, ..., n_k$ is the largest number that can not be represented as a linear combination of $n_1, ..., n_k$ with non-negative integer coefficients.

We will assume that all the lengths l_j are integers and denote by $t = 0$ the instant, when the initial quasi-particle meets the first vertex. Let us call the time after which the number of points ceases to grow a *stabilization time* and denote it by t_{st}. It is easy to see, that, if the lengths $l_1, \cdots, l_s$ are not relatively prime, then

$$t_{st}(l_1, \ldots, l_s) = t_{st}(\frac{l_1}{\mathrm{gcd}, (l_1 \cdots, l_s)}, \cdots, \frac{l_s}{\mathrm{gcd}(l_1, \cdots, l_s)}) \cdot \mathrm{gcd}(l_1, \cdots, l_s)$$

so we can consider only relatively prime lengths.

Proposition 3. *The stabilization time for star graph with two edges with relatively prime lengths can be represented in the following form:*

$$t_{st} = 2(max(l_1, l_2) + Fr(l_1, l_2)).$$

A similar formula is valid for the star graph with three edges, but not for more complicated graphs. We don't know any explicit formula for the stabilization time for an arbitrary graph.

3.3 Case of the Polynomial Growth of $m(t)$

Asymptotics of $N(t)$ Suppose that the number of geodesics $m(t)$ grows polynomially as $t \to \infty$. Such a situation takes place for the manifolds with not very complicated geodesic flow (see examples below). Note that there are popular classes of such manifolds; in particular, so called uniformly secure ones (the manifold is called uniformly secure, if there exists an integer R, such that for arbitrary pair of points all geodesics, connecting these points, can be blocked by an R-point obstacle; see, e.g. [12]). In this case the number of quasi-particles grows in a sub-exponential way.

Theorem 7. *Let $m(t) = c_0 t^\gamma (1 + O(t^{-\varepsilon}))$, $\gamma > 0, \varepsilon > 0$. Let the set of lengths L_j, l_j be linearly independent over $\mathbf{Q}$ (i.e. any finite subset of lengths in linearly independent). Then*

$$\log N(t) = (\gamma + 1) \left(\frac{c_0 \Gamma(\gamma + 1) \zeta(\gamma + 1)}{\gamma^\gamma} \right)^{\frac{1}{\gamma+1}} t^{\frac{\gamma}{\gamma+1}} (1 + o(1)) \tag{13}$$

Here $\Gamma(x)$ and $\zeta(x)$ are the Γ-function and the Riemann ζ-function.

Remark 8. This result can be generalized for the case of sets of lengths L, containing linearly dependent over $\mathbf{Q}$ finite subsets. Namely, it is sufficient to suppose that there exists a finite subset $L_0 \subset L$, such that $L \backslash L_0$ is linearly independent. If there is no such subset L_0, the equality (13) must be replaced by the inequality—the right hand side defines the upper bound for N.

Examples 1. Hybrid space, obtained by gluing an interval to a cylinder Let us consider a circular cylinder $M_1 \subset \mathbf{R}^3$ with a length of a circle equal to b with the induced metric. Let the points q_1 and q_2 lie on a ruling of the cylinder at the distance a from each other. We glue a segment to these two points. The lengths of geodesics are equal to $\sqrt{(nb)^2 + a^2}$, $n \in \mathbf{N}$; for almost all real a and b they are linearly independent over $\mathbf{Q}$ and

$$\log N(t) = \sqrt{\frac{2}{3b}}\ \pi\ t^{\frac{1}{2}}(1 + o(1))$$

Remark 9. Note that for $b = 1$ the latter formula coincides with the Hardy—Ramanujan formula for the number $N(t)$ of partitions of an integer t.

2. Hybrid space obtained by gluing an interval to a flat torus
Let us consider a flat 2D torus M_1 with fundamental cycles of lengths a and b. Let us consider a fundamental rectangle with sides a, b and take points $q_1 = (0, 0)$, $q_2 = (c, d)$ in it. We glue a segment to these points.

Then for almost all a, b, c, d

$$\log N(t) = 3\left(\frac{5\pi}{8ab}\zeta(3)\right)^{\frac{1}{3}} t^{\frac{2}{3}}(1 + o(1)).$$

The analogous formula for $3D$ torus with fundamental cycles a, b, c has the form

$$\log N(t) = 4\left(\frac{\pi}{3abc}\zeta(4)\right)^{\frac{1}{4}} t^{\frac{3}{4}}(1 + o(1)).$$

3.4 Exponential Growth of m

Finally, we suppose that the function $m(t)$ grows exponentially. Note that this case is typical for geodesic flows with positive topological entropy: if M is a compact Riemannian manifold, then the topological entropy H of the geodesic flow equals

$$H = \lim_{t\to\infty} \frac{1}{t} \log \int_{M\times M} m_{x,y}(t) dx dy,$$

where $m_{x,y}(t)$ denotes the number of geodesics with the lengths at most t, connecting the points x and y. Moreover, if M does not have conjugate points (this is the case, for example, for compact surfaces of constant negative curvature), then for arbitrary pair of points x, y

$$H = \lim_{t\to\infty} \frac{1}{t} \log m_{x,y}(t).$$

([12, 13])

Theorem 8. *Let* $\log m(t) = Ht(1 + t^{-\varepsilon})$, $\varepsilon > 0$. *Let the set of lengths* L_j, l_j *be linearly independent over* $\mathbf{Q}$ *(i.e. any finite subset of lengths in linearly independent). Then*

$$\log N(t) = Ht(1 + o(1)). \tag{14}$$

Remark 10. This result can be generalized for the case of sets of lengths L, containing linearly dependent over $\mathbf{Q}$ finite subsets. Namely, it is sufficient to suppose that there exists a finite subset $L_0 \subset L$, such that $L \backslash L_0$ is linearly independent. If there is no such subset L_0, the equality (14) must be replaced by the inequality—the right hand side defines the upper bound for N.

Sometimes one can obtain a more detailed estimate.

Proposition 4. *Let $m(t) = e^{Ht}(b_0 + b_1(Ht)^{-1} + O(t^{-2}))$, If there exists a finite subset $L_0 \subset L$ such that all numbers from $L \setminus L_0$ are linearly independent over $\mathbf{Q}$, then $\forall j = 0, \ldots, m$*

$$\log N(t) = Ht + 2\sqrt{b_0 Ht} + \left(\frac{b_1}{2} - \frac{3}{4}\right) \log t + O(1).$$

In the opposite case, this equality turns into upper bound.

4 Abstract Prime Numbers Distributions

The main steps in the proofs of the previous theorems are following: the problem of the computation of $N(t)$ can be reduced to the problem of the analytic number theory. Namely, consider an arithmetic semigroup

$$G = \oplus_{j \in J} \mathbf{Z}_+,$$

where J is a countable set, and a homomorphism $\rho : G \to \mathbf{R}_+$, such that for arbitrary $t \in R_+$ the set of elements $g \in G$ with $\rho(g) \leq t$ is finite. We can identify elements $j \in J$ with the corresponding generators of $\mathbf{Z}_+$. Consider two functions

$$m(t) = \sharp\{j \in J | \rho(j) \leq t\}, \quad N(t) = \sharp\{g \in G, \rho(g) \leq t\}.$$

The direct (inverse) problem of abstract primes distribution is the following question. If one knows the asymtotics of N (m), how to compute the asymptotics of m (N)?

Remark 11. If J is the set of primes and $\rho(j)$=$\log j$, then $m(\log t)$ is the distribution function of primes and $N(t)$ is the integral part of t.

If J is the set of integers and $\rho(j) = j$ then $N(t)$ is the number of partitions of integer t and $m(t) = t$.

The result of the previous sections follow from the results, obtained in [14–17]; these results give the solution of the inverse problem of abstract primes distribution for the cases of polynomial and exponential growths of $m(t)$.

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Normal Equation Generated from Helmholtz System: Nonlocal Stabilization by Starting Control and Properties of Stabilized Solutions

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Abstract. We consider the semilinear normal parabolic equation (NPE) corresponding to the 3D Helmholtz system with periodic boundary conditions. First, we recall the main definitions and results associated with the NPE including a result on stabilization to zero of the solution for NPE with arbitrary initial condition by starting control. The main content of the paper is to study properties of stabilized solution of NPE.

Keywords: Semilinear normal parabolic equation
3D Helmholtz system · Stabilisation theory · Navier-Stokes equations

1 Introduction

The purpose of this article is to develop the theory of the nonlocal stabilization by starting control for the normal parabolic equations generated from a three-dimensional Helmholtz system with periodic boundary conditions. Such theory has been constructed first for NPE associated with the Burgers equation (see [3,6]), and after that for NPE associated with the Helmholtz system in [7].

Recall the setting of non-local stabilization problem by starting control for NPE corresponding to the 3D Helmholtz system: Let $0 < a_j < b_j < 2\pi, j = 1, 2, 3$ be fixed. Given a divergence free initial condition $y_0(x)$ for NPE associated with the 3D Helmholtz system, find a divergence free starting control $u_0(x)$ supported in $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{T}^3$ such that the solution $y(t, x)$ of NPE with initial condition $y_0 + u_0$ satisfies the inequality

$$\|y(t,\cdot)\|_{L_2(\mathbb{T}^3)} \leq \alpha \|y_0 + u_0\|_{L_2(\mathbb{T}^3)} e^{-t}, \quad \forall t > 0, \tag{1}$$

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with some $\alpha > 1$. Here, $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{N})^3$ is the torus that is the definition area for periodic functions.

The problem formulated above has been solved in [7]. Namely, it has been proved that the NPE with arbitrary initial condition y_0 can be stabilized by starting control in the form

$$u_0(x) = Fy_0 - \lambda u(x), \tag{2}$$

where Fy_0 is a certain feedback control with feedback operator F constructed by some technic of local stabilization theory (see [4,5]), $\lambda > 0$ is a constant, depending on y_0, and u is a universal function, depending only on a given arbitrary parallelepiped $[a_1, b_1]\times[a_2, b_2]\times[a_3, b_3] \subset \mathbb{T}^3$, which contains the support of control u_0. The following estimate is the key one for the proof of the stabilization result:

$$\int_{\mathbb{T}^3} ((\mathbf{S}(t,x;u), \nabla)\operatorname{curl}^{-1}\mathbf{S}(t,x;u), \mathbf{S}(t,x;u))dx > 3\beta e^{-18t}, \qquad \forall\, t \geq 0, \tag{3}$$

where $S(t, x; u)$ is the solution of the Stokes equation with initial condition u and $\beta > 0$ is some constant.

The proof of the estimate (3) is quite complicated, and has been made in [3,6,7].

In this paper, we develop the stabilization result indicated above. The point is that our strategic aim is to extend this result from NPE to the Helmholtz system. In order to do this, we need to establish some additional properties of the stabilized solution. The first of them is the following estimate of a stabilized solution $y(t, x; v)$, where $v = y_0 + u_0 = y_0 + Fy_0 - \lambda u$:

$$\|y(t,\cdot;v)\| \leq \frac{\|v\|e^{-t}}{1 + \frac{\beta}{16}\|v\|(1 - e^{-16t})}, \tag{4}$$

where β is the constant from (3). Henceforth, we use the notation

$$\|\cdot\| = \|\cdot\|_{L_2(\mathbb{T}^3)} \tag{5}$$

The bound (4) implies one important corollary. To formulate it, recall first the following property of NPE: [1]

There exists $r_0 > 0$ such that for each initial condition y_0 with $\|y_0\| \leq r_0$, the solution $y(t, x; y_0)$ of NPE satisfies

$$\|y(t,\cdot;y_0)\| \leq 2\|y_0\|e^{-t}, \quad \text{as} \quad t \to \infty. \tag{6}$$

An important corollary of the bound (4) is formulated as follows:

There exists instant t_0 such that the solution of NPE with initial condition $v = y_0 + u_0$ satisfies $\|y(t_0,\cdot;v)\| \leq r_0$, and t_0 does not depend on $\|v\|$.

[1] Of course, many other equations including the Helmholtz system possess this property.

This property as well as the estimate (4) are obtained in the Sect. 4.

In Sect. 2, we remind the definitions and some facts concerning NPE associated with the 3D Helmholtz system. Section 3 is devoted to the formulation of the main stabilization result, the formulation of the result related to the estimate (3) and the definition of feedback operator F from (2). At last, in Subsect. 3.4, we present the derivation of the main nonlocal stabilization result from the bound (3) in more details than in [7]. This simplifies our considerations in the Sect. 4.

2 Semilinear Parabolic Equation of Normal Type

In this section, we recall the basic information on parabolic equations of normal type corresponding to 3D Helmholtz system: their derivation, an explicit formula for solutions, a theorem on the existence and uniqueness of solution for normal parabolic equations and the structure of their dynamics. These results have been obtained in [1,2].

2.1 Navier-Stokes Equations

Let us consider the 3D Navier-Stokes system

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)v + \nabla p(t,x) = 0,\ \mathrm{div}\ v = 0, \tag{7}$$

with periodic boundary conditions

$$v(t,\ldots,x_i,\ldots) = v(t,\ldots,x_i+2\pi,\ldots),\ i=1,2,3, \tag{8}$$

and initial condition

$$v(t,x)|_{t=0} = v_0(x), \tag{9}$$

where $t \in \mathbb{R}_+$, $x = (x_1,x_2,x_3) \in \mathbb{R}^3$, $v(t,x) = (v_1,v_2,v_3)$ is the velocity vector field of fluid flow, ∇p is the gradient of pressure, Δ is the Laplace operator, $(v,\nabla)v = \sum_{j=1}^3 v_j \partial_{x_j} v$. The periodic boundary conditions (8) mean that Navier-Stokes equations (7) and initial conditions (9) are defined on torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$.

For each $m \in \mathbb{Z}_+ = \{j \in \mathbb{Z} : j \geq 0\}$ we define the space

$$V^m = V^m(\mathbb{T}^3) = \{v(x) \in (H^m(\mathbb{T}^3))^3 : \mathrm{div}\ v = 0, \int_{\mathbb{T}^3} v(x)dx = 0\} \tag{10}$$

where $H^m(\mathbb{T}^3)$ is the Sobolev space.

It is well-known, that the nonlinear term $(v,\nabla)v$ in problem (7)–(9) satisfies relation

$$\int_{\mathbb{T}^3} (v(t,x),\nabla)v(t,x)\cdot v(t,x)dx = 0.$$

Therefore, multiplying (7) scalarly by v in $L_2(\mathbb{T}^3)$, integrating by parts by x, and then integrating by t, we obtain the well-known energy estimate

$$\int_{\mathbb{T}^3} |v(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{T}^3} |\nabla_x v(\tau,x)|^2 dx d\tau \leq \int_{\mathbb{T}^3} |v_0(x)|^2 dx, \tag{11}$$

which allows to prove the existence of a weak solution for (7)–(9). But, as is well-known, scalar multiplication of (7) by v in $V^1(\mathbb{T}^3)$ does not result into an analog of estimate (11). Nevertheless, such kind of expression will be useful for us. More precisely, we will consider the scalar product in V^0 of Helmholtz equations by its unknown vector field (which is equivalent).

2.2 Helmholtz Equations

Using problem (7)–(9) for fluid velocity v, let us derive the similar problem for the curl of velocity

$$\omega(t,x) = \operatorname{curl} v(t,x) = (\partial_{x_2} v_3 - \partial_{x_3} v_2, \partial_{x_3} v_1 - \partial_{x_1} v_3, \partial_{x_1} v_2 - \partial_{x_2} v_1), \tag{12}$$

from it.

It is well-known from vector analysis, that

$$(v,\nabla)v = \omega \times v + \nabla \frac{|v|^2}{2}, \tag{13}$$

$$\operatorname{curl}(\omega \times v) = (v,\nabla)\omega - (\omega,\nabla)v, \text{ if } \operatorname{div} v = 0, \operatorname{div} \omega = 0, \tag{14}$$

where $\omega \times v = (\omega_2 v_3 - \omega_3 v_2, \omega_3 v_1 - \omega_1 v_3, \omega_1 v_2 - \omega_2 v_1)$ is the vector product of ω and v, and $|v|^2 = v_1^2 + v_2^2 + v_3^2$. Substituting (13) into (7) and applying the curl operator to both sides of the obtained equation, taking into account (12), (14) and formula $\operatorname{curl} \nabla F = 0$, we obtain the Helmholtz equations

$$\partial_t \omega(t,x) - \varDelta\omega + (v,\nabla)\omega - (\omega,\nabla)v = 0, \tag{15}$$

with initial conditions

$$\omega(t,x)|_{t=0} = \omega_0(x) := \operatorname{curl} v_0(x), \tag{16}$$

and periodic boundary conditions.

2.3 Derivation of Normal Parabolic Equations (NPE)

Using decomposition into Fourier series

$$v(x) = \sum_{k\in\mathbb{Z}^3} \hat{v}(k) e^{i(k,x)}, \ \hat{v}(k) = (2\pi)^{-3} \int_{\mathbb{T}^3} v(x) e^{-i(k,x)} dx, \tag{17}$$

where $(k,x) = k_1 \cdot x_1 + k_2 \cdot x_2 + k_3 \cdot x_3$, $k = (k_1, k_2, k_3)$, and the well-known formula $\operatorname{curl}\operatorname{curl} v = -\varDelta v$, if $\operatorname{div}\, v = 0$, we see that inverse operator to curl is well-defined on space V^m and is given by the formula

$$\operatorname{curl}^{-1} \omega(x) = i \sum_{k\in\mathbb{Z}^3} \frac{k \times \hat{\omega}(k)}{|k|^2} e^{i(k,x)}. \tag{18}$$

Therefore, operator curl : $V^1 \mapsto V^0$ realizes an isomorphism of the spaces, thus, a sphere in V^1 for (7)–(9) is equivalent to a sphere in V^0 for the problem (15)–(16).

Let us denote the nonlinear term in Helmholtz system by B:

$$B(\omega) = (v, \nabla)\omega - (\omega, \nabla)v, \tag{19}$$

where v can be expressed in terms of ω using (18).

Multiplying (19) scalarly by $\omega = (\omega_1, \omega_2, \omega_3)$ and integrating by parts, we obtain the expression

$$(B(\omega), \omega)_{V_0} = -\int_{\mathbb{T}^3} \sum_{j,k=1}^{3} \omega_j \partial_j v_k \omega_k dx, \tag{20}$$

which, generally speaking, is not zero. Hence, the energy estimate for solutions of 3D Helmholtz system is not fulfilled. In other words, operator B allows the decomposition

$$B(\omega) = B_n(\omega) + B_\tau(\omega), \tag{21}$$

where vector $B_n(\omega)$ is orthogonal to the sphere $\Sigma(\|\omega\|_{V^0}) = \{u \in V^0 : \|u\|_{V^0} = \|\omega\|_{V^0}\}$ at the point ω, and vector B_τ is tangent to $\Sigma(\|\omega\|_{V^0})$ at ω. In general, both terms in (21) are not equal to zero. Since the presence of B_n, and not of B_τ, prevents the fulfillments of the energy estimate, it is plausible that just B_n generates the possible singularities in the solution. Therefore, it seems reasonable to omit the B_τ term in Helmholtz system and study first the analog of (15) where the nonlinear operator $B(\omega)$ is replaced with $B_n(\omega)$. We will call the resulting equations the normal parabolic equations.

Let us now derive the NPE corresponding to (15)–(16).

Since summand $(v, \nabla)\omega$ in (19) is tangential to vector ω, the normal part of operator B is defined by the summand $(\omega, \nabla)v$. We shall seek it in the form $\Phi(\omega)\omega$, where Φ is the unknown functional, which can be found from equation

$$\int_{\mathbb{T}^3} \Phi(\omega)\omega(x) \cdot \omega(x) dx = \int_{\mathbb{T}^3} (\omega(x), \nabla)v(x) \cdot \omega(x) dx. \tag{22}$$

According to (22),

$$\Phi(\omega) = \begin{cases} \int_{\mathbb{T}^3} (\omega(x), \nabla) \operatorname{curl}^{-1} \omega(x) \cdot \omega(x) dx / \int_{\mathbb{T}^3} |\omega(x)|^2 dx, \ \omega \neq 0, \\ \hfill 0, \ \omega \equiv 0, \end{cases} \tag{23}$$

where $\operatorname{curl}^{-1} \omega(x)$ is defined in (18).

Thus, we arrive at the following system of normal parabolic equations corresponding to Helmholtz equations (15):

$$\partial_t \omega(t, x) - \Delta\omega - \Phi(\omega)\omega = 0, \ \operatorname{div} \omega = 0, \tag{24}$$

$$\omega(t, \dots, x_i, \dots) = \omega(t, \dots, x_i + 2\pi, \dots), \ i = 1, 2, 3, \tag{25}$$

where Φ is the functional defined in (23).

Further, we study problem (24), (25) with initial condition (16).

2.4 Explicit Formula for Solution of NPE

In this subsection, we remind the explicit formula for the NPE solution.

Lemma 2.1 *Let* $\mathbf{S}(t,x;\omega_0)$ *be the solution of the following Stokes system with periodic boundary conditions:*

$$\partial_t z - \Delta z = 0, \text{ div } z = 0; \tag{26}$$

$$z(t,...,x_i+2\pi,...) = z(t,x), \quad i = 1,2,3; \tag{27}$$

$$z(0,x) = \omega_0, \tag{28}$$

i.e. $\mathbf{S}(t,x;\omega_0) = z(t,x)$. [2] *Then, the solution of problem* (24) *with periodic boundary conditions and initial condition* (16) *has the form*

$$\omega(t,x;\omega_0) = \frac{\mathbf{S}(t,x;\omega_0)}{1-\int_0^t \Phi(\mathbf{S}(\tau,\cdot;\omega_0))d\tau}. \tag{29}$$

One can see the proof of this Lemma in [1,2].

2.5 Properties of the Functional $\boldsymbol{\Phi(u)}$

Let $s \in \mathbb{R}$. By definition, the Sobolev space $H^s(\mathbb{T}^3)$ is the space of periodic real distributions with finite norm

$$\|z\|^2_{H^s(\mathbb{T}^3)} \equiv \|z\|^2_s = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |k|^{2s}|\widehat{z}(k)|^2 < \infty, \tag{30}$$

where $\widehat{z}(k)$ are the Fourier coefficients[3] of the function z.

We shall use the following generalization of the spaces (10) of solenoidal vector fields:

$$V^s \equiv V^s(\mathbb{T}^3) = \left\{v(x) \in (H^s(\mathbb{T}^3))^3 : \text{ div } v(x) = 0, \int_{\mathbb{T}^3} v(x)\,dx = 0\right\}, \quad s \in \mathbb{R}. \tag{31}$$

Lemma 2.2 *Let* $\Phi(u)$ *be the functional* (23)*. Then, there exists a constant* $c > 0$ *such that, for every* $u \in V^{3/2}$,

$$|\Phi(u)| \le c\|u\|_{3/2}. \tag{32}$$

[2] Note that because of periodic boundary conditions the Stokes system should not contain the pressure term ∇p.

[3] Strictly speaking, one needs to add $|\widehat{z}(0)|^2$ to the right-hand side of (30). However, we did not do this, since starting from (31) it will be assumed that this Fourier coefficient is zero.

Proof. The estimate

$$|\Phi(u)| \le \frac{\|u\|^2_{L_3(\mathbb{T}^3)}\|\nabla \operatorname{curl}^{-1} u\|_{L_3(\mathbb{T}^3)}}{\|u\|_0^2} \le c\frac{\|u\|^3_{1/2}}{\|u\|_0^2} \le c\frac{\|u\|_0^2\|u\|_{3/2}}{\|u\|_0^2} = c\|u\|_{3/2} \tag{33}$$

follows from definition (23), Sobolev's embedding theorem (according to which $H^{1/2}(\mathbb{T}^3) \subset L_3(\mathbb{T}^3)$) and the interpolation inequality $\|v\|^3_{1/2} \le c\|v\|_0^2\|v\|_{3/2}$.

Lemma 2.3 *Let Φ be the functional (23). For any $\beta < 1/2$, there exists a constant $c_1 > 0$ such that*

$$\left|\int_0^t \Phi(S(\tau; y_0))\, d\tau\right| \le c_1\|y_0\|_{-\beta} \tag{34}$$

for any $y_0 \in V^{-\beta}(\mathbb{T}^3)$ and $t > 0$. Here, $S(t; y_0)$ is the solution operator of problem (26), (27), (28). [4]

Proof. Using (32) and the representation of the solution of problem (26), (27), (28) in terms of the Fourier series, we see that

$$\left|\int_0^t \Phi(S(\tau; y_0))\, d\tau\right| \le c\int_0^t e^{-\tau/2}\left(\sum_{k\ne 0}(|\widehat{y}_0(k)|^2|k|^{-2\beta})|k|^{3+2\beta}e^{-(k^2-1)\tau}\right)^{1/2} d\tau, \tag{35}$$

where $\widehat{y}_0(k)$ are the Fourier coefficients of the function y_0. The solution $\widehat{\rho} = \widehat{\rho}(t)$ of the extremal problem

$$f(t,\rho) = \rho^{3+2\beta}e^{-(\rho^2-1)t} \to \max, \quad \rho \ge 1,$$

is given by $\widehat{\rho}(t) = \sqrt{\frac{3+2\beta}{2t}}$. We also have

$$f(t,\widehat{\rho}(t)) = \begin{cases} \left(\frac{3+2\beta}{2t}\right)^{\frac{3+2\beta}{2}} e^{-(3+2\beta-2t)/2}, & t \le \frac{3+2\beta}{2}, \\ 1, & t \ge \frac{3+2\beta}{2}. \end{cases} \tag{36}$$

Substituting (36) into (35), we arrive at (34), the result required.

Remark 2.1 In view of Lemma 2.3, the functional on the left of (34) is well-defined for $y_0 \in V^{-\beta}(\mathbb{T}^3)$ with $\beta < 1/2$. In particular, Lemma 2.3 and (29) show that the solution of problem (24), (25), (16) is well-defined for any initial data $y_0 \in V^0$ and is infinitely differentiable in these variables for each $x \in \mathbb{T}^3$ and $t \in (0, T)$ where T depends on initial condition y_0.

In the following two sections we will justify the choice of the space V^0 as the phase space of the corresponding dynamical system.

[4] Here and below we use for brevity notation $S(t; y_0)$ instead of $S(t, \cdot; y_0)$

2.6 Unique Solvability of NPE

Let $Q_T = (0, T) \times \mathbb{T}^3$, $T > 0$ or $T = \infty$. The following space of solutions for NPE will be used:

$$V^{1,2(-1)}(Q_T) = L_2(0, T; V^1) \cap H^1(0, T; V^{-1})$$

We look for solutions $\omega(t, x; \omega_0)$ satisfying

Condition 2.1 *If the initial condition $\omega_0 \in V^0 \backslash \{0\}$ and the solution $\omega(t, x; \omega_0) \in V^{1,2(-1)}(Q_T)$, then $\omega(t, \cdot, \omega_0) \neq 0$, $\forall t \in [0, T]$.*

Theorem 1. *For each $\omega_0 \in V^0$ there exists $T > 0$ such that there exists unique solution $\omega(t, x; \omega_0) \in V^{1,2(-1)}(Q_T)$ of the problem (24), (25), (16) satisfying Condition (2.1)*

Theorem 2. *The solution $\omega(t, x; \omega_0) \in V^{1,2(-1)}(Q_T)$ of the problem (24), (25), (16) depends continuously on initial condition $\omega_0 \in V^0$.*

One can see the proof of these Theorems in [2].

2.7 Structure of Dynamical Flow for NPE

We will use $V^0(\mathbb{T}^3) \equiv V^0$ as the phase space for problem (24), (25), (16).

Definition 2.1 *The set $M_- \subset V^0$ of ω_0, such that the corresponding solution $\omega(t, x; \omega_0)$ of problem (24), (25), (16) satisfies inequality*

$$\|\omega(t, \cdot; \omega_0)\|_0 \leq \alpha \|\omega_0\|_0 e^{-t}, \qquad \forall t > 0,$$

is called the set of stability. Here, $\alpha > 1$ is a fixed number depending on $\|\omega_0\|_0$.

Definition 2.2 *The set $M_+ \subset V^0$ of ω_0, such that the corresponding solution $\omega(t, x; \omega_0)$ exists only on a finite time interval $t \in (0, t_0)$ and blows up at $t = t_0$, is called the set of explosions.*

Definition 2.3 *The set $M_g \subset V^0$ of ω_0, such that the corresponding solution $\omega(t, x; \omega_0)$ exists for time $t \in \mathbb{R}_+$, and $\|\omega(t, x; \omega_0)\|_0 \to \infty$, as $t \to \infty$, is called the set of growing.*

Lemma 2.4 *(see [2]) Sets M_-, M_+, M_g are not empty, and $M_- \cup M_+ \cup M_g = V^0$.*

2.8 On a Geometrical Structure of Phase Space

Let us define the following subsets of unit sphere: $\Sigma = \{v \in V^0 : \ \|v\|_0 = 1\}$ in the phase space V^0:

$$A_-(t) = \{v \in \Sigma : \int_0^t \Phi(S(\tau, v))d\tau \le 0\}, \quad A_- = \cap_{t \ge 0} A_-(t),$$

$$B_+ = \Sigma \setminus A_- \equiv \{v \in \Sigma : \ \exists t_0 > 0 \ \int_0^{t_0} \Phi(S(\tau, v))d\tau > 0\},$$

$$\partial B_+ = \{v \in \Sigma : \ \forall t > 0 \int_0^t \Phi(S(\tau, v))d\tau \le 0 \quad \text{and } \exists t_0 > 0 : \ \int_0^{t_0} \Phi(S(\tau, v))d\tau = 0\}.$$

We introduce the following function on sphere Σ:

$$B_+ \ni v \to b(v) = \max_{t \ge 0} \int_0^t \Phi(S(\tau, v))d\tau. \tag{37}$$

Evidently, $b(v) > 0$ $b(v) \to 0$ as $v \to \partial B_+$. Let us define the map $\Gamma(v)$:

$$B_+ \ni v \to \Gamma(v) = \frac{1}{b(v)} v \in V^0. \tag{38}$$

It is clear that $\|\Gamma(v)\|_0 \to \infty$, as $v \to \partial B_+$. The set $\Gamma(B_+)$ divides V^0 into two parts:

$$V^0_- = \{v \in V^0 : \ [0, v] \cap \Gamma(B_+) = \emptyset\},$$
$$V^0_+ = \{v \in V^0 : \ [0, v) \cap \Gamma(B_+) \ne \emptyset\}.$$

Let $B_+ = B_{+,f} \cup B_{+,\infty}$ where

$$B_{+,f} = \{v \in B_+ : \ \text{max in (37) is achived at } t < \infty\},$$

$$B_{+,\infty} = \{v \in B_+ : \ \text{max in (37) is not achived at } t < \infty\}.$$

Theorem 3. *(see [2])* $M_- = V^0_-, \ M_+ = V^0_+ \cup B_{+,f}, \ M_g = B_{+,\infty}$

3 Stabilization of Solution for NPE by Starting Control

3.1 Formulation of the Main Result on Stabilization

We consider the semilinear parabolic equations (24):

$$\partial_t y(t, x) - \Delta y(t, x) - \Phi(y)y = 0 \tag{39}$$

with periodic boundary condition

$$y(t, \dots, x_i + 2\pi, \dots) = y(t, x), \ i = 1, 2, 3, \tag{40}$$

and an initial condition

$$y(t,x)|_{t=0} = y_0(x) + u_0(x). \tag{41}$$

Here, Φ is the functional defined in (23), $y_0(x) \in V^0(\mathbb{T}^3)$ is an arbitrary given initial datum and $u_0(x) \in V^0(\mathbb{T}^3)$ is a control. The phase space V^0 is defined in (10).

We assume that $u_0(x)$ is supported on $[a_1,b_1] \times [a_2,b_2] \times [a_3,b_3] \subset \mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$:

$$\operatorname{supp} u_0 \subset [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3] \tag{42}$$

(here we treat the torus $\mathbb{T}^3$ as the cube $[0,2\pi]^3$ where 0 and 2π are identified).

Our goal is to find for every given $y_0(x) \in V^0(\mathbb{T}^3)$ a control $u_0 \in V^0(\mathbb{T}^3)$ satisfying (42) such that there exists unique solution $y(t,x;y_0+u_0)$ of (39)–(41), and this solution satisfies the estimate

$$\|y(t,\cdot;y_0+u_0)\|_0 \leq \alpha \|y_0+u_0\|_0 e^{-t}, \quad \forall t > 0, \tag{43}$$

with a certain $\alpha > 1$.

By Definition 2.1 of the set of stability M_- inclusion $y_0 \in M_-$ implies estimate (43) with $u_0 = 0$. Therefore, the formulated problem is reach of content only if $y_0 \in V^0 \setminus M_- = M_+ \cup M_g$. Note that, without loss of generality, the last inclusion can be changed on $y_0 \in V^{1/2} \setminus M_-$. Indeed, in virtue of explicit formula (29) the solution $y(t,\cdot;y_0)$ of NPE belongs to $C^\infty(\mathbb{T}^3)$ for arbitrary small $t > 0$. Hence, if $y_0 \in V^0$ we can shift on small t, take $y(t,\cdot;y_0)$ as initial condition and apply to it stabilization construction.

The following main theorem holds:

Theorem 4. *Let $y_0 \in V^{1/2} \setminus M_-$ be given. Then, there exists a control $u_0 \in V^0 \cap (L_3(\mathbb{T}^3))^3$ satisfying (42) such that there exists a unique solution $y(t,x;y_0+u_0)$ of (39)–(41), and this solution satisfies bound (43) with a certain $\alpha > 1$.*

Below we will indicate the main steps of this theorem proof.

3.2 Formulation of the Main Preliminary Result

To rewrite condition (42) in more convenient form, let us first perform the change of variables in (39)–(41):

$$\tilde{x}_i = x_i - \frac{a_i+b_i}{2}, i = 1,2,3,$$

and denote

$$\begin{aligned}
\tilde{y}(t,\tilde{x}) &= y\left(t, \tilde{x}_1 + \frac{a_1+b_1}{2}, \tilde{x}_2 + \frac{a_2+b_2}{2}, \tilde{x}_3 + \frac{a_3+b_3}{2}\right), \\
\tilde{y}_0(\tilde{x}) &= y_0\left(\tilde{x}_1 + \frac{a_1+b_1}{2}, \tilde{x}_2 + \frac{a_2+b_2}{2}, \tilde{x}_3 + \frac{a_3+b_3}{2}\right), \\
\tilde{u}_0(\tilde{x}) &= u_0\left(\tilde{x}_1 + \frac{a_1+b_1}{2}, \tilde{x}_2 + \frac{a_2+b_2}{2}, \tilde{x}_3 + \frac{a_3+b_3}{2}\right).
\end{aligned} \tag{44}$$

Then, substituting (44) into relations (39)–(41), (43) and omitting the tilde sign, these relations remain unchanged, while inclusion (42) transforms into

$$\operatorname{supp} u_0 \subset [-\rho_1, \rho_1] \times [-\rho_2, \rho_2] \times [-\rho_3, \rho_3], \tag{45}$$

where $\rho_i = \dfrac{b_i - a_i}{2} \in (0, \pi)$, $i = 1, 2, 3$.

Below we consider stabilization problem (39)–(41), (43) with condition (45) instead of (42).

We look for a starting control $u_0(x)$ of the form

$$u_0(x) = u_1(x) - \lambda u(x), \tag{46}$$

where the component $u_1(x)$ and the constant $\lambda > 0$ will be defined later and the main component $u(x)$ is defined as follows. For given $\rho_1, \rho_2, \rho_3 \in (0, \pi)$ we choose $p \in \mathbb{N}$ such that

$$\frac{\pi}{p} \le \rho_i,\ i = 1, 2, 3, \tag{47}$$

and denote by $\chi_{\frac{\pi}{p}}(\alpha)$ the characteristic function of interval $(-\frac{\pi}{p}, \frac{\pi}{p})$:

$$\chi_{\frac{\pi}{p}}(\alpha) = \begin{cases} 1, \ |\alpha| \le \frac{\pi}{p}, \\ 0, \ \frac{\pi}{p} < |\alpha| \le \pi. \end{cases} \tag{48}$$

Then, we set

$$u(x) = \frac{\tilde{u}}{\|\tilde{u}\|_0} \quad \text{with } \tilde{u}(x) = \operatorname{curl}\operatorname{curl}(\chi_{\frac{\pi}{p}}(x_1)\chi_{\frac{\pi}{p}}(x_2)\chi_{\frac{\pi}{p}}(x_3) w(px_1, px_2, px_3), 0, 0), \tag{49}$$

where

$$w(x_1, x_2, x_3) = \sum_{\substack{i,j,k=1 \\ i<j, k \ne i,j}}^{3} a_k(1 + \cos x_k)(\sin x_i + \frac{1}{2}\sin 2x_i)(\sin x_j + \frac{1}{2}\sin 2x_j), \tag{50}$$

$a_1, a_2, a_3 \in \mathbb{R}$.

Proposition 3.1 *The vector field $u(x)$ defined in (47)–(50) possesses the following properties:*

$$u(x) \in V^0(\mathbb{T}^3) \cap (L_\infty(\mathbb{T}^3)^3), \qquad \mathit{supp}\, u \subset ([-\rho, \rho])^3, \quad \|u\|_0 = 1. \tag{51}$$

Proof. For each $j = 1, 2, 3$ function $w(x_1, x_2, x_3)$ defined in (50) and $\partial_j w$ equal to zero at $x_j = \pm\pi$. That is why using notations $\chi_{\frac{\pi}{p}}(x) = \chi_{\frac{\pi}{p}}(x_1)\chi_{\frac{\pi}{p}}(x_2)\chi_{\frac{\pi}{p}}(x_3)$, $w(px) = w(px_1, px_2, px_3)$ we obtain

$$\operatorname{curl}(\chi_{\frac{\pi}{p}}(x)(w(px), 0, 0)) = p\chi_{\frac{\pi}{p}}(x)(0, \partial_3 w(px), -\partial_2 w(px)) \in (H^1(\mathbb{T}^3))^3, \tag{52}$$

$$u(x) = p^2\chi_{\frac{\pi}{p}}(x)(-\partial_{22}w(px) - \partial_{33}w(px), \partial_{12}w(px), -\partial_{13}w(px)) \in (H^0(\mathbb{T}^3))^3. \tag{53}$$

Evidently, vector field (53) satisfies also the inclusion

$$u(x) \in (L_\infty(\mathbb{T}^3))^3$$

Applying the operator div to vector field (53) and performing direct calculations in the space of distributions, it follows that div $u(x) = 0$. Hence, $u(x) \in V^0(\mathbb{T}^3)$. The second and third relations in (51) are evident.

Let us consider the boundary value problem for the system of three heat equations

$$\partial_t \mathbf{S}(t,x;u) - \Delta \mathbf{S}(t,x;u) = 0, \qquad \mathbf{S}(t,x)|_{t=0} = u(x), \tag{54}$$

with periodic boundary condition. (Since by Proposition 3.1 div $u(x) = 0$, we get that div $S(t,x;u) = 0$ for $t > 0$, and therefore system (54) in fact is equal to the Stokes system.)

The following theorem is true:

Theorem 5. *For each $\rho := \pi/p \in (0,\pi)$ the function $u(x)$ defined in (49)–(50) by a natural number p satisfying (47) and the characteristic function (48) satisfies the estimate:*

$$\int_{T^3} ((\mathbf{S}(t,x;u),\nabla)\,\mathrm{curl}^{-1}\,\mathbf{S}(t,x;u),\mathbf{S}(t,x;u))dx > 3\beta e^{-18t}, \qquad \forall\, t \geq 0, \tag{55}$$

with a positive constant β.

The proof of Theorem 5 is given in [7]. This theorem is the main and the most difficult step of Theorem's 4 proof.

3.3 Intermediate Control

To avoid certain difficulties with the proof of Theorem 4, we have to include additional control that eliminates some Fourier coefficients in given initial condition y_0 of our stabilization problem. We will use the techniques developed in local stabilization theory (see [4,5] and references therein).

Let us consider the following decomposition of the phase space: $V^0 = V_+ \oplus V_-$, where

$$V_+ = \{v \in V^0 : v(x) = \sum_{0<|k|^2<18} v_k e^{i(k,x)}, v_k \in \mathbb{C}^3, k \cdot v_k = 0, v_{-k} = \overline{v_k}\}, \tag{56}$$

where, recall $\overline{v}_k$ means complex conjugation of v_k, $k = (k_1,k_2,k_3), |k|^2 = k_1^2 + k_2^2 + k_3^2$, and

$$V_- = V^0 \ominus V_+. \tag{57}$$

Theorem 6. *There exists a linear feedback operator F,*

$$F : V^0(\mathbb{T}^3) \mapsto V^1_{00}(\varOmega) := \{y(x) \in V^1(\mathbb{T}^3) : supp\, y \subset \varOmega\}, \tag{58}$$

where $\varOmega = \{x \in ([-\rho, \rho])^3 : |x|^2 \leq \rho^2\} \subset \mathbb{T}^3$, $\rho = \pi/p, p \in \mathbb{N} \setminus \{1\}$, such that for every $y \in V^0$

$$y + Fy \in V_- \quad , \tag{59}$$

where V_- is the subset of V^0 defined in (56)–(57)

The proof of Theorem 6 see in [7].

The component $u_1(x)$ of the control (46) is defined as follows:

$$u_1 = Fy_0 \tag{60}$$

where F is the operator defined in Theorem 6, and y_0 is initial condition from stabilization problem (39), (40), (41).

3.4 Proof of the Stabilization Result

In this subsection we prove Theorem 4 using Theorems 5, 6. We take control (46) as a desired one where vector-functions $u_1(x)$, $u(x)$ are defined in (60) and (49)–(50), respectively, and $\lambda \gg 1$ is a parameter.

In virtue of explicit formula (29) for solution of NPE, in order to prove the desired result it is enough to choose parameter λ in such way that the function

$$1 - \int_0^t \varPhi(\mathbf{S}(\tau, \cdot; y_0 + u_1 - \lambda u))d\tau, \tag{61}$$

for each $t > 0$ is bounded from below by a positive constant independent of t. For this aim we estimate the function $-\varPhi(\mathbf{S}(t, \cdot; z_0 - \lambda u))$ where $z_0 = y_0 + u_1 = y_0 + Fy_0$. In virtue of Theorem 6 $z_0 \in V_-$, where V_- is the subset of the phase space defined in (56), i.e. Fourier coefficients of z_0 satisfy the condition

$$\hat{z}_0(k) = 0 \text{ for } |k|^2 = k_1^2 + k_2^2 + k_3^2 < 18. \tag{62}$$

Let us denote the nominator of functional $\varPhi$ defined in (23) as follows:

$$\varPsi(y_1, y_2, y_3) = \int_{T^3} ((y_1, \nabla)\text{curl}^{-1} y_2, y_3)dx, \quad \varPsi(y) = \varPsi(y, y, y) \tag{63}$$

and prove the estimate

$$-\varPsi(\mathbf{S}(t, \cdot; z_0 - \lambda u)) > 2\beta\lambda^3 e^{-18t} \quad \text{for } \lambda \gg 1, \tag{64}$$

where β is the positive constant from Theorem 5.

According to Theorem 5,

$$\varPsi(\mathbf{S}(t, \cdot; u)) \geq 3\beta e^{-18t}, \beta > 0. \tag{65}$$

From definition (63) of Ψ,

$$
\begin{aligned}
&-\Psi(\mathbf{S}(t,z_0-\lambda u))=\lambda^3\Psi(\mathbf{S}(t,u))-\lambda^2(\Psi(\mathbf{S}(t,u),\mathbf{S}(t,u),\mathbf{S}(t,z_0))+\\
&\Psi(\mathbf{S}(t,u),\mathbf{S}(t,z_0),\mathbf{S}(t,u))+\Psi(\mathbf{S}(t,z_0),\mathbf{S}(t,u),\mathbf{S}(t,u)))+\\
&\lambda(\Psi(\mathbf{S}(t,u),\mathbf{S}(t,z_0),\mathbf{S}(t,z_0))+\Psi(\mathbf{S}(t,z_0),\mathbf{S}(t,u),\mathbf{S}(t,z_0))+\\
&\Psi(\mathbf{S}(t,z_0),\mathbf{S}(t,z_0),\mathbf{S}(t,u)))-\Psi(\mathbf{S}(t,z_0)).
\end{aligned}
\tag{66}
$$

In virtue of Sobolev embedding theorem and definition (63) we obtain

$$
\begin{aligned}
&|\Psi(y_1,y_2,y_3)|\le\|y_1\|_{L_3(\mathbb{T}^3)}\|\nabla\mathrm{curl}^{-1}y_2\|_{L_3}\|y_3\|_{L_3}\le\\
&c_2'\|y_1\|_{L_3(\mathbb{T}^3)}\|y_2\|_{L_3}\|y_3\|_{L_3}\le c_2\|y_1\|_{V^{1/2}}\|y_2\|_{V^{1/2}}\|y_3\|_{V^{1/2}}.
\end{aligned}
\tag{67}
$$

Besides, (67) implies

$$
|\Psi(y_1,y_2,y_3)|\le c_2\|y_1\|_{L_\infty(\mathbb{T}^3)}\|y_2\|_{L_\infty}\|y_3\|_{L_\infty}.
\tag{68}
$$

Using (65)–(68) we get inequalities

$$
\begin{aligned}
&-\Psi(\mathbf{S}(t,z_0-\lambda u))>3\beta\lambda^3e^{-18t}-c\left(\lambda^2\|\mathbf{S}(t,u)\|^2_{V^{1/2}}\|\mathbf{S}(t,z_0)\|_{V^{1/2}}+\right.\\
&\left.\lambda\|\mathbf{S}(t,u)\|_{V^{1/2}}\|\mathbf{S}(t,z_0)\|^2_{V^{1/2}}+\|\mathbf{S}(t,z_0)\|^3_{V^{1/2}}\right)
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
&-\Psi(\mathbf{S}(t,z_0-\lambda u))>3\beta\lambda^3e^{-18t}-c\left(\lambda^2\|\mathbf{S}(t,u)\|^2_{L_\infty}\|\mathbf{S}(t,z_0)\|_{V^{1/2}}+\right.\\
&\left.\lambda\|\mathbf{S}(t,u)\|_{L_\infty}\|\mathbf{S}(t,z_0)\|^2_{V^{1/2}}+\|\mathbf{S}(t,z_0)\|^3_{V^{1/2}}\right).
\end{aligned}
\tag{70}
$$

Let us show that for a fixed small enough $t_0>0$

$$
\|\mathbf{S}(t,u)\|_{V^{1/2}}\le\frac{e^{t_0-1/4}}{\sqrt{2}t_0^{1/4}}\|u\|_{V^0}e^{-t},\qquad\forall t>t_0.
\tag{71}
$$

In virtue of (54) for a fixed small enough $t_0>0$

$$
\|\mathbf{S}(t,u)\|^2_{V^{1/2}}=\sum_{\mathbb{Z}^3\setminus\{0\}}|u_k|^2|k|e^{-2k^2t_0}e^{-2k^2(t-t_0)}\le\|\mathbf{S}(t_0,u)\|^2_{V^{1/2}}e^{-2(t-t_0)}\ \forall t>t_0.
\tag{72}
$$

Besides, taking into account that for $f(y,t_0)=ye^{-2y^2t_0}$ the solution $\hat{y}$ of extremal problem

$$
\max_{y\ge 2}f(y,t_0)\quad\text{is}\quad\hat{y}=\frac{1}{2\sqrt{t_0}}\quad\text{and}\quad f(\hat{y},t_0)=\frac{e^{-1/2}}{2\sqrt{t_0}},
$$

it follows that

$$
\|\mathbf{S}(t_0,u)\|^2_{V^{1/2}}=\sum_{\mathbb{Z}^3\setminus\{0\}}|u_k|^2|k|e^{-2k^2t_0}\le\frac{e^{-1/2}}{2\sqrt{t_0}}\|u\|^2_{V^0}.
\tag{73}
$$

Now (71) follows from (72), (73).

Using the maximum principle for a heat equation we obtain

$$\|\mathbf{S}(t,u)\|_{L_\infty} \le \|u\|_{L_\infty},\ t\in(0,t_0). \tag{74}$$

In virtue of (62),

$$\|\mathbf{S}(t,z_0)\|_{V^{1/2}} \le \|z_0\|_{V^{1/2}} e^{-18t}. \tag{75}$$

Relations (69), (71), (75) imply

$$-\Psi(\mathbf{S}(t,z_0-\lambda u)) > 3\beta\lambda^3 e^{-18t} - c_5(\lambda^2 e^{-20t}+\lambda e^{-37t}+e^{-54t}) > 2\beta\lambda^3 e^{-18t}, \tag{76}$$

for $t\ge t_0$, and relations (70), (74), (75) imply (76) for $t\in[0,t_0]$. We can take $t_0=\frac{1}{16}$ or less. This completes the proof of estimate (64).

The denominator of $\Phi(\mathbf{S}(\tau,\cdot;z_0-\lambda u))$ is positive, i.e.

$$\int_{\mathbb{T}^3} |\mathbf{S}(t,z_0-\lambda u)|^2 dx > 0. \tag{77}$$

Bounds (64), (77) imply that the function (61) is more than 1, which completes the proof of Theorem 4.

4 Some Development of Stabilization Theory of Solution for NPE

In this section we develop the theory whose expozition began in the previous section. Our main goal is to prepare the necessary basis that will give us the opportunity to generalize the stabilization theory from NPE to Helmholtz system.

4.1 One Corollary of Theorem 5

Our first step is to rewrite estimate (64) in a more convenient form.

Proposition 4.1 *Let $\Psi(y_1,y_2,y_3)$ be functional (63), and $u(x)$ be control (49), (50). Then for each $z_0=y_0+Fy_0$ where $y_0\in V^{1/2}\setminus M_-$ there exists $\lambda_0=\lambda_0(\|y_0\|_{V^{1/2}})$ such that for any $\lambda>\lambda_0$*

$$\frac{-\Psi(\mathbf{S}(t,\cdot;z_0-\lambda u))}{\|\mathbf{S}(t,\cdot;z_0-\lambda u)\|^3_{V^0}} > \beta e^{-15t}, \quad \forall t\ge 0, \tag{78}$$

where β is the positive constant from Theorem 5.

Proof. In what follows, we will use notation $\|\cdot\|=\|\cdot\|_{V^0}$. Since $\|u\|=1$, we have

$$2=2\|u\|^3 \ge 2\left(\|u-\frac{z_0}{\lambda}\|-\|\frac{z_0}{\lambda}\|\right)^3 = \|u-\frac{z_0}{\lambda}\|^3+B, \tag{79}$$

where

$$B = \|u - \frac{z_0}{\lambda}\|^3 - 6\|u - \frac{z_0}{\lambda}\|^2\|\frac{z_0}{\lambda}\| + 6\|u - \frac{z_0}{\lambda}\|\|\frac{z_0}{\lambda}\|^2 - 2\|\frac{z_0}{\lambda}\|^3 =$$
$$\|u - \frac{z_0}{\lambda}\|^2(\|u - \frac{z_0}{\lambda}\| - 6\|\frac{z_0}{\lambda}\|) + 6\|\frac{z_0}{\lambda}\|^2(\|u - \frac{z_0}{\lambda}\| - \frac{1}{3}\|\frac{z_0}{\lambda}\|) \geq \tag{80}$$
$$\|u - \frac{z_0}{\lambda}\|^2(1 - 7\|\frac{z_0}{\lambda}\|) + 6\|\frac{z_0}{\lambda}\|^2(1 - \frac{4}{3}\|\frac{z_0}{\lambda}\|) > 0,$$

for $\lambda > 7\|z_0\|$. Hence $2 > \|u - \frac{z_0}{\lambda}\|^3$. Applying this inequality to the right side of (64) and dividing the obtained inequality by $\lambda^3\|u - \frac{z_0}{\lambda}\|^3$, it follows that

$$\frac{-\Psi(\mathbf{S}(t, \cdot; z_0 - \lambda u))}{\|z_0 - \lambda u\|^3} > \beta e^{-18t}. \tag{81}$$

Similarly to estimate (72)

$$\|\mathbf{S}(t, z_0 - \lambda u)\|_{V^0}^2 = \sum_{\mathbb{Z}^3 \setminus \{0\}} |z_{0,k} - \lambda u_k|^2 e^{-2k^2 t} \leq \|z_0 - \lambda u\|_{V^0}^2 e^{-2t}, \ \forall t > 0. \tag{82}$$

Dividing $-\Psi(\mathbf{S}(t, \cdot; z_0 - \lambda u))$ on $\|\mathbf{S}(t, z_0 - \lambda u)\|_{V^0}^3$, taking into account (82) and after that (81), we obtain

$$\frac{-\Psi(\mathbf{S}(t, \cdot; z_0 - \lambda u))}{\|\mathbf{S}(t, z_0 - \lambda u)\|^3} > \frac{-\Psi(\mathbf{S}(t, \cdot; z_0 - \lambda u))}{\|z_0 - \lambda u\|^3 e^{-3t}} > \beta e^{-15t}. \tag{83}$$

This proves the estimate (78).

4.2 The Estimate of Stabilized Solution

The main aim of this subsection is to prove the following.

Theorem 7. *Let u, z_0, y_0, λ_0 be as in Proposition 4.1, and $y(t, x; v)$ be the solution of problem (39), (40), (41) with $v := y_0 + u_0 = z_0 - \lambda u$, $\lambda > \lambda_0$. Then, $y(t, x; v)$ satisfies the inequality*

$$\|y(t, \cdot; v)\| \leq \frac{\|v\|e^{-t}}{1 + \frac{\beta}{16}\|v\|(1 - e^{-16t})}, \tag{84}$$

where $\beta > 0$ is the constant from Theorem 5.

Proof. Multiplying equation (39) by $y(t, x; v)$ scalarly in V^0 and taking into account the notations (23), (63) of functionals Φ, Ψ, we get after simple transformation

$$\frac{1}{2}\partial_t\|y(t, v)\|^2 + \|y_x(t, v)\|^2 - \Psi(y(t, v)) = 0. \tag{85}$$

Dividing this equality by $\|y(t, v)\|^3$, we obtain that

$$\frac{\partial_t\|y(t, v)\|}{\|y(t, v)\|^2} + \frac{\|y_x(t, v)\|^2}{\|y(t, v)\|^3} = \frac{\Psi(y(t, v))}{\|y(t, v)\|^3} \tag{86}$$

Using notation $z(t) = 1/\|y(t,v)\|$ and taking into account Friedrichs inequality $\|y_x\| \le \|y\|$ we derive from (86), the inequality

$$-\partial_t z(t) + z(t) = \frac{\Psi(y(t,v))}{\|y(t,v)\|^3} - \frac{1}{\|y(t,v)\|}\left(\frac{\|y_x(t,v)\|^2}{\|y(t,v)\|^2} - 1\right) \le \frac{\Psi(y(t,v))}{\|y(t,v)\|^3} \quad (87)$$

Let us transform the right side of (87). In virtue of explicit formula (29)

$$\|y(t,v)\|^3 \le \frac{\|\mathbf{S}(t,v)\|^3}{\left(1 - \int_0^t \frac{\Psi(\mathbf{S}(t,v))}{\|\mathbf{S}(t,v)\|^2}\right)^3}, \quad -\Psi(y(t,v)) = \frac{-\Psi(\mathbf{S}(t,v))}{\left(1 - \int_0^t \frac{\Psi(\mathbf{S}(t,v))}{\|\mathbf{S}(t,v)\|^2}\right)^3} \quad (88)$$

Dividing the second relation of (88) on the first one we obtain that

$$\frac{-\Psi(y(t,v))}{\|y(t,v)\|^3} \ge \frac{-\Psi(\mathbf{S}(t,v))}{\|\mathbf{S}(t,v)\|^3}. \quad (89)$$

Inequalities (87), (89), (78) imply the estimate

$$\partial_t z(t) - z(t) \ge \beta e^{-15t}, \quad \text{or} \quad z(t) \ge e^t\left(\frac{1}{\|v\|} + \frac{\beta}{16}(1 - e^{-16t})\right). \quad (90)$$

Passing from $z(t)$ to $\|y(t,v)\|$ we obtain the inequality (84).

The estimate (84) has important corollaries. To show them, recall the following.

Lemma 4.1 *There exists $r_0 > 0$ such that for each $y_0 \in B_{r_0} = \{y \in V^0 : \|y\| \le r_0\}$ the solution $y(t,x;y_0)$ of problem (39), (40) with initial condition $y(t,x)|_{t=0} = y_0$ satisfies*

$$\|y(t,\cdot;y_0)\| \le c_0 e^{-t}, \qquad as \quad t \to \infty. \quad (91)$$

Proof. In virtue of the explicit formula (29) and Lemma 2.3

$$\|y(t,\cdot;y_0)\| \le \frac{\|\mathbf{S}(t,\cdot;y_0)\|}{1 - \int_0^t \Phi(\mathbf{S}(\tau,y_0))d\tau} \le \frac{\|y_0\|e^{-t}}{1 - c_1\|y_0\|}, \quad (92)$$

where c_1 is the constant from the bound (34). So if we take in (92) $\|y_0\| = r_0 = \frac{1}{2c_1}$ we get the bound (91) with $c_0 = 1/c_1$.

Lemma 4.1 implies that, in order to solve the stabilization problem (39), (40), (41), it is enough to check that at some instant t_0 its solution $y(t_0,x;y_0+u_0)$ belongs to the ball B_{r_0}.

Now we prove the following corollary of Theorem 7.

Proposition 4.2 *Let $y(t,x;v)$ be the solution of problem (39), (40), (41) from Theorem 7. Then, there exists $t_0 > 0$ independent of $\|v\|$ such that $y(t_0,\cdot;v) \in B_{r_0}$ where $r_0 = \frac{1}{2c_1}$ is defined in Lemma 4.1.*

Proof. Removing the summand 1 in the denominator from the bound (84), we obtain

$$\|y(t,\cdot;v)\| \leq \frac{16e^{-t}}{\beta(1-e^{-16t})}. \tag{93}$$

Taking into account this bound we should find t satisfying

$$\frac{16e^{-t}}{\beta(1-e^{-16t})} \leq r_0. \tag{94}$$

Making the change $x = e^{-t}$ and as in Lemma 4.1 $r_0 = \frac{1}{2c_1}$, we reduce the deal to solution of the equation

$$F(x) := \beta x^{16} + 32c_1 x - \beta = 0. \tag{95}$$

Since $F(0) = -\beta < 0,\ F(1) = 32c_1 > 0$, and $F'(x) > 0$ for $x \in (0,1)$, there exist unique $x_0 \in (0,1)$ satisfying $F(x_0) = 0$. Therefore, for $t_0 = \ln\frac{1}{x_0}$, we get that $y(t_0,\cdot;v) \in B_{r_0}$ with $r_0 = \frac{1}{2c_1}$.

Thus, Proposition 4.2 establishes some kind of duality between the behavior of NPS solution with initial conditions from sets of explosions M_+ and the stability M_- defined in Subsect. 2.7. Indeed, if $y_0 \in M_+$ then the norm of solution $\|y(t,\cdot;y_0)\|$ blows up at some finite instant $\hat{t}$, i.e. $\|y(t,\cdot;y_0)\| \to \infty$ as $t \nearrow \hat{t}$. In case $v \in M_-$ by Proposition 4.2 there exist fixed not big instant t_0 and quantity r_0 such that $\|y(t_0,\cdot;v)\| \leq r_0$ for initial condition v with arbitrary big $\|v\|$.

Applications of Theorem 7 and Proposition 4.2 will be given in future publications.

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